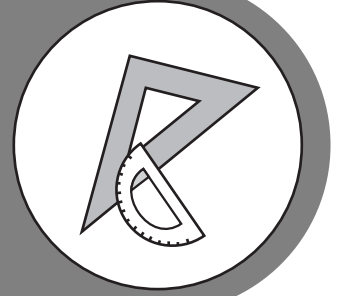


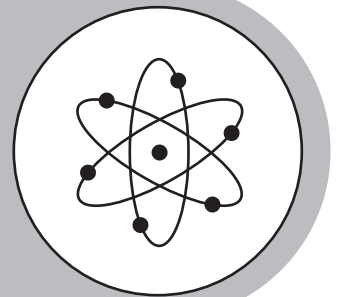
# GPS ALGEBRA



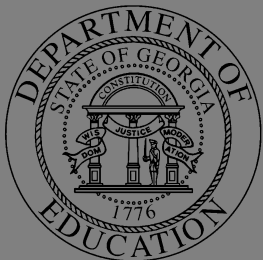
## Study



## Guide



## Georgia End-Of-Course Tests





# TABLE OF CONTENTS

<b>INTRODUCTION .....</b>	<b>5</b>
<b>HOW TO USE THE STUDY GUIDE.....</b>	<b>6</b>
<b>OVERVIEW OF THE EOCT .....</b>	<b>8</b>
<b>PREPARING FOR THE EOCT.....</b>	<b>9</b>
Study Skills .....	9
Time Management .....	10
Organization.....	10
Active Participation.....	11
Test-taking Strategies.....	11
Suggested Strategies to Prepare for the EOCT .....	12
Suggested Strategies the Day before the EOCT .....	13
Suggested Strategies the Morning of the EOCT .....	13
Top 10 Suggested Strategies during the EOCT .....	14
<b>TEST CONTENT .....</b>	<b>15</b>
Studying the Content Standards and Topics .....	16
Unit 1: The Chance of Winning.....	17
Unit 2: Algebra Investigations .....	40
Unit 3: Function Families .....	58
Unit 4: Algebra in Context.....	73
Unit 5: Quadratics and Complex Numbers .....	94
Unit 6: Putting the Pieces Together .....	118
<b>APPENDICES</b>	
<b>APPENDIX A:</b> EOCT Sample Overall Study Plan Sheet.....	145
<b>APPENDIX B:</b> Blank Overall Study Plan Sheet .....	146
<b>APPENDIX C:</b> EOCT Sample Daily Study Plan Sheet.....	147
<b>APPENDIX D:</b> Blank Daily Study Plan Sheet .....	148

**This Page Is Intentionally  
Left Blank.**

## INTRODUCTION

This study guide is designed to help students prepare to take the Georgia End-of-Course Test (EOCT) for *GPS Algebra*. This study guide provides information about the EOCT, tips on how to prepare for it, and some suggested strategies students can use to perform their best.

**What is the EOCT?** The EOCT program was created to improve student achievement through effective instruction and assessment of the standards in the Georgia Performance Standards (GPS). The EOCT program helps to ensure that all Georgia students have access to a rigorous curriculum that meets high performance standards. The purpose of the EOCT is to provide diagnostic data that can be used to enhance the effectiveness of schools' instructional programs.

The Georgia End-of-Course Testing program is a result of the A+ Educational Reform Act of 2000, O.C.G.A. §20-2-281. This act requires that the Georgia Department of Education create end-of-course assessments for students in grades 9 through 12 for the following core high school subjects:

### Mathematics

- Mathematics I: Algebra/Geometry/Statistics
- Mathematics II: Geometry/Algebra II/Statistics

--OR--

- GPS Algebra
- GPS Geometry

### Social Studies

- United States History
- Economics/Business/Free Enterprise

### Science

- Biology
- Physical Science

### English Language Arts

- Ninth Grade Literature and Composition
- American Literature and Composition

**Getting started:** The HOW TO USE THE STUDY GUIDE section on page 6 outlines the contents in each section, lists the materials you should have available as you study for the EOCT, and suggests some steps for preparing for the *GPS Algebra EOCT*.

## HOW TO USE THE STUDY GUIDE

This study guide is designed to help you prepare to take the *GPS Algebra EOCT*. It will give you valuable information about the EOCT, explain how to prepare to take the EOCT, and provide some opportunities to practice for the EOCT. The study guide is organized into three sections. Each section focuses on a different aspect of the EOCT.

The OVERVIEW OF THE EOCT section on page 8 gives information about the test: dates, testing time, question format, and number of questions that will be on the *GPS Algebra EOCT*. This information can help you better understand the testing situation and what you will be asked to do.

The PREPARING FOR THE EOCT section that begins on page 9 provides helpful information on study skills and general test-taking skills and strategies. It explains how to prepare before taking the test and what to do during the test to ensure the best test-taking situation possible.

The TEST CONTENT section that begins on page 15 explains what the *GPS Algebra EOCT* specifically measures. When you know the test content and how you will be asked to demonstrate your knowledge, it will help you be better prepared for the EOCT. This section also contains some sample EOCT test questions, helpful for gaining an understanding of how a standard may be tested.

With some time, determination, and guided preparation, you will be better prepared to take the *GPS Algebra EOCT*.



### GET IT TOGETHER

In order to make the most of this study guide, you should have the following:

#### Materials:

- \* This study guide
- \* Pen or pencil
- \* Highlighter
- \* Paper

#### Resources:

- \* Classroom notes
- \* Mathematics textbook
- \* A teacher or other adult

#### Study Space:

- \* Comfortable (but not too comfortable)
- \* Good lighting
- \* Minimal distractions
- \* Enough work space

#### Time Commitment:

- \* When are you going to study?
- \* How long are you going to study?

#### Determination:

- \* Willingness to improve
- \* Plan for meeting goals



## ***SUGGESTED STEPS FOR USING THIS STUDY GUIDE***

- 1** Familiarize yourself with the structure and purpose of the study guide. (You should have already read the INTRODUCTION and HOW TO USE THE STUDY GUIDE. Take a few minutes to look through the rest of the study guide to become familiar with how it is arranged.)
  
- 2** Learn about the test and expectations of performance. (Read OVERVIEW OF THE EOCT.)
  
- 3** Improve your study skills and test-taking strategies. (Read PREPARING FOR THE EOCT.)
  
- 4** Learn what the test will assess by studying each unit and the strategies for answering questions that assess the standards in the unit. (Read TEST CONTENT.)
  
- 5** Answer the sample test question at the end of each lesson. Check your answer against the answer given to see how well you did. (See TEST CONTENT.)

## OVERVIEW OF THE EOCT

Good test takers understand the importance of knowing as much about a test as possible. This information can help you determine how to study and prepare for the EOCT and how to pace yourself during the test. The box below gives you a snapshot of the *GPS Algebra EOCT* and other important information.



### THE EOCT AT A GLANCE

**Administration Dates:**

The EOCT has three primary annual testing dates: once in the spring, once in the summer, and once in the winter. There are also mid-month, online tests given in August, September, October, November, February, and March, as well as retest opportunities within the year.

**Administration Time:**

Each EOCT is composed of two sections, and students are given 60 minutes to complete each section. There is also a short stretch break between the two sections of the test.

**Question Format:**

All the questions on the EOCT are multiple-choice.

**Number of Questions:**

Each section of the *GPS Algebra EOCT* contains 31 questions; there are a total of 62 questions on the *GPS Algebra EOCT*.

**Impact on Course Grade:**

For students in grade 10 or above beginning the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 85% and the EOCT score 15%. For students in grade 9 beginning the 2011–2012 school year and later, the final grade in each course is calculated by weighing the course grade 80% and the EOCT score 20%. A student must have a final grade of at least 70 to pass the course and to earn credit toward graduation.

If you have additional administrative questions regarding the EOCT, please visit the Georgia Department of Education Web site at [www.doe.k12.ga.us](http://www.doe.k12.ga.us), see your teacher, or see your school test coordinator.



## PREPARING FOR THE EOCT



### WARNING!

You cannot prepare for this kind of test in one night. Questions will ask you to apply your knowledge, not list specific facts. Preparing for the EOCT will take time, effort, and practice.



To do your best on the *GPS Algebra EOCT*, it is important that you take the time necessary to prepare for this test and develop those skills that will help you take the EOCT.

First, you need to make the most of your classroom experiences and test preparation time by using good **study skills**. Second, it is helpful to know general **test-taking strategies** to ensure that you will achieve your best score.

### Study Skills



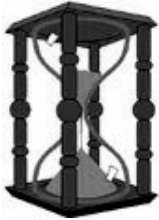
#### A LOOK AT YOUR STUDY SKILLS

Before you begin preparing for this test, you might want to consider your answers to the following questions. You may write your answers here or on a separate piece of paper.

1. How would you describe yourself as a student?  
Response: \_\_\_\_\_
2. What are your study skills strengths and/or weaknesses as a student?  
Response: \_\_\_\_\_
3. How do you typically prepare for a mathematics test?  
Response: \_\_\_\_\_
4. Are there study methods you find particularly helpful? If so, what are they?  
Response: \_\_\_\_\_
5. Describe an ideal study situation (environment).  
Response: \_\_\_\_\_
6. Describe your actual study environment.  
Response: \_\_\_\_\_
7. What can you change about the way you study to make your study time more productive?  
Response: \_\_\_\_\_

Effective study skills for preparing for the EOCT can be divided into three categories:

- ◆ **Time Management**
- ◆ **Organization**
- ◆ **Active Participation**



## Time Management

Do you have a plan for preparing for the EOCT? Often students have good intentions for studying and preparing for a test, but without a plan, many students fall short of their goals. Here are some strategies to consider when developing your study plan:

- ◆ Set realistic goals for what you want to accomplish during each study session and chart your progress.
- ◆ Study during your most productive time of the day.
- ◆ Study for reasonable amounts of time. Marathon studying is not productive.
- ◆ Take frequent breaks. Breaks can help you stay focused. Doing some quick exercises (e.g., sit-ups or jumping jacks) can help you stay alert.
- ◆ Be consistent. Establish your routine and stick to it.
- ◆ Study the most challenging test content first.
- ◆ For each study session, build in time to review what you learned in your last study session.
- ◆ Evaluate your accomplishments at the end of each study session.
- ◆ Reward yourself for a job well done.

## Organization

You don't want to waste your study time. Searching for materials, trying to find a place to study, and debating what and how to study can all keep you from having a productive study session. Get organized and be prepared. Here are a few organizational strategies to consider:



- ◆ Establish a study area that has minimal distractions.
- ◆ Gather your materials in advance.
- ◆ Develop and implement your study plan (see Appendices A–D for sample study plan sheets).

## Active Participation



Students who actively study will learn and retain information longer. Active studying also helps you stay more alert and be more productive while learning new information. What is active studying? It can be anything that gets you to interact with the material you are studying. Here are a few suggestions:

- ◆ Carefully read the information and then **DO** something with it. Mark the important points with a highlighter, circle it with a pen, write notes on it, or summarize the information in your own words.
- ◆ Ask questions. As you study, questions often come into your mind. Write them down and actively seek the answers.
- ◆ Create sample test questions and answer them.
- ◆ Find a friend who is also planning to take the test and quiz each other.

## Test-taking Strategies

There are many test-taking strategies that you can use before and during a test to help you have the most successful testing situation possible. Below are a few questions to help you take a look at your test-taking skills.




### A LOOK AT YOUR TEST-TAKING SKILLS

As you prepare to take the EOCT, you might want to consider your answers to the following questions. You may write your answers here or on your own paper.

1. How would you describe your test-taking skills?  
Response: \_\_\_\_\_
2. How do you feel when you are taking a test?  
Response: \_\_\_\_\_
3. List the strategies that you already know and use when you are taking a test.  
Response: \_\_\_\_\_
4. List test-taking behaviors you use that contribute to your success when preparing for and taking a test.  
Response: \_\_\_\_\_
5. What would you like to learn about taking tests?  
Response: \_\_\_\_\_

## Suggested Strategies to Prepare for the EOCT

 **Learn from the past.** Think about your daily/weekly grades in your mathematics classes (past and present) to answer the following questions:

- In which specific areas of mathematics were you or are you successful?

Response: \_\_\_\_\_


- Is there anything that has kept you from achieving higher scores?


Response: \_\_\_\_\_

- What changes should you implement to achieve higher scores?

Response: \_\_\_\_\_

Before taking the EOCT, work toward removing or minimizing any obstacles that might stand in the way of performing your best. The test preparation ideas and test-taking strategies in this section are designed to help guide you to accomplish this.

 **Be prepared.** The best way to perform well on the EOCT is to be prepared. In order to do this, it is important that you know what standards/skills will be measured on the **GPS Algebra EOCT** and then practice understanding and using those standards/skills. The TEST CONTENT section of this study guide is designed to help you understand the specific standards that are on the **GPS Algebra EOCT** and give you suggestions for how to study the standards that will be assessed. Take the time to read through this material and follow the study suggestions. You can also ask your math teacher for any suggestions he or she might offer on preparing for the EOCT.

 **Start now.** Don't wait until the last minute to start preparing. Begin early and pace yourself. By preparing a little bit each day, you will retain the information longer and increase your confidence level. Find out when the EOCT will be administered, so you can allocate your time appropriately.

## Suggested Strategies the Day before the EOCT

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

- ✓ **Review what you learned from this study guide.**
  1. Review the general test-taking strategies discussed in the TOP 10 SUGGESTED STRATEGIES DURING THE EOCT on page 14.
  2. Review the content information discussed in the TEST CONTENT section beginning on page 15.
  3. Focus your attention on the main topic, or topics, that you are most in need of improving.
  
- ✓ **Take care of yourself.**
  1. Try to get a good night's sleep. Most people need an average of eight hours, but everyone's sleep needs are different.
  2. Don't drastically alter your routine. If you go to bed too early, you might lie in bed thinking about the test. You want to get enough sleep so you can do your best.

## Suggested Strategies the Morning of the EOCT



**Eat a good breakfast.** Choose foods high in protein for breakfast (and for lunch if the test is given in the afternoon). Some examples of foods high in protein are peanut butter, meat, and eggs. Protein gives you long-lasting, consistent energy that will stay with you through the test to help you concentrate better. Avoid foods high in sugar content. It is a misconception that sugar sustains energy—after an initial boost, sugar will quickly make you more tired and drained. Also, don't eat too much. A heavy meal can make you feel tired. So think about what you eat before the test.



**Dress appropriately.** If you are too hot or too cold during the test, it can affect your performance. It is a good idea to dress in layers, so you can stay comfortable, regardless of the room temperature, and keep your mind on the EOCT.







**Arrive for the test on time.** Racing late into the testing room can cause you to start the test feeling anxious. You want to be on time and prepared.

# TOP 10

## Suggested Strategies during the EOCT

These general test-taking strategies can help you do your best during the EOCT.

- 1 Focus on the test.**  Try to block out whatever is going on around you. Take your time and think about what you are asked to do. Listen carefully to all the directions.
- 2 Budget your time.**  Be sure that you allocate an appropriate amount of time to work on each question on the test.
- 3 Take a quick break if you begin to feel tired.** To do this, put your pencil down, relax in your chair, and take a few deep breaths. Then, sit up straight, pick up your pencil, and begin to concentrate on the test again. Remember that each test section is only 60 minutes.
- 4 Use positive self-talk.** If you find yourself saying negative things to yourself such as “I can’t pass this test,” it is important to recognize that you are doing this. Stop and think positive thoughts such as “I prepared for this test, and I am going to do my best.” Letting the negative thoughts take over can affect how you take the test and your test score.
- 5 Mark in your test booklet.**  Mark key ideas or things you want to come back to in your test booklet. Remember that only the answers marked on your answer sheet will be scored.
- 6 Read the entire question and the possible answer choices.** It is important to read the entire question so you know what it is asking. Read each possible answer choice. Do not mark the first one that “looks good.”
- 7 Use what you know.**  Draw on what you have learned in class, from this study guide, and during your study sessions to help you answer the questions.
- 8 Use content domain-specific strategies to answer the questions.** In the TEST CONTENT section, there are a number of specific strategies that you can use to help improve your test performance. Spend time learning these helpful strategies, so you can use them while taking the test.
- 9 Think logically.** If you have tried your best to answer a question but you just aren’t sure, use the process of elimination. Look at each possible answer choice. If it doesn’t seem like a logical response, eliminate it. Do this until you’ve narrowed down your choices. If this doesn’t work, take your best educated guess. It is better to mark something down than to leave it blank.
- 10 Check your answers.**  When you have finished the test, go back and check your work.

### A WORD ON TEST ANXIETY

It is normal to have some stress when preparing for and taking a test. It is what helps motivate us to study and try our best. Some students, however, experience anxiety that goes beyond normal test “jitters.” If you feel you are suffering from test anxiety that is keeping you from performing at your best, please speak to your school counselor, who can direct you to resources to help you address this problem.

## TEST CONTENT

Up to this point in this study guide, you have been learning various strategies on how to prepare for and take the EOCT. This section focuses on what will be tested. It also includes sample questions that will let you apply what you have learned in your classes and from this study guide.

This section of the study guide will help you learn and review the various mathematical concepts that will appear on the ***GPS Algebra EOCT***. Since *mathematics* is a broad term that covers many different topics, the state of Georgia has divided it into two major areas of knowledge called **content strands**. The content strands are broad categories. Each content strand is broken down into big ideas. These big ideas are called **content standards** or just standards. Each content strand contains standards that cover different ideas related to the content strand. Each question on the EOCT measures an individual standard within a content strand.

The two content strands within the ***GPS Algebra EOCT*** are Algebra (which includes Number and Operations) and Data Analysis and Probability. These are important for several reasons. Together, they cover the major skills and concepts needed to understand and solve mathematical problems. These skills have many practical applications in the real world. Another more immediate reason that the content strands are important has to do with test preparation. The best way to prepare for any test is to study and know the material measured on the test.

This study guide is organized in six **units** that cover the content standards outlined by unit on the GPS Algebra curriculum map. Each unit is presented by topic rather than by specific strand or standard (although those are listed at the beginning of each unit and are integral to each topic). The more you understand about the topics in each unit, the greater your chances of getting a good score on the EOCT.

## Studying the Content Standards and Topics (Units 1 through 6)

You should be familiar with many of the content standards and topics that follow. It makes sense to spend more time studying the content standards and topics that you think may cause you problems. Even so, do not skip over any of them. The TEST CONTENT section has been organized into six units. Each unit is organized by the following features:

- **Introduction:** an overview of what will be discussed in the unit
- **Key Standards:** information about the specific standards that will be addressed  
(NOTE: The names of the standards may not be the exact names used by the Georgia Department of Education.)

- **Main Topics:** the broad subjects covered in the unit

**Each Main Topic includes:**

- **Key Ideas:** definitions of important words and ideas as well as descriptions, examples, and steps for solving problems
- **Review Examples:** problems with solutions showing possible ways to answer given questions
- **EOCT Practice Items:** sample multiple-choice questions similar to test items on the *GPS Algebra EOCT* with answers provided

With some time, determination, and guided preparation, you will be better prepared to take the *GPS Algebra EOCT*.



## **Unit 1: The Chance of Winning**

In this unit, students will calculate probabilities based on angles and area models, compute simple permutations and combinations, calculate and display summary statistics, and calculate expected values. Students will also use simulations and statistics as tools to answer difficult theoretical probability questions.

### **KEY STANDARDS**

**MM1D1. Students will determine the number of outcomes related to a given event.**

- a. Apply the addition and multiplication principles of counting.
- b. Calculate and use simple permutations and combinations.

**MM1D2. Students will use the basic laws of probability.**

- a. Find the probabilities of mutually exclusive events.
- b. Find probabilities of dependent events.
- c. Calculate conditional probabilities.
- d. Use expected value to predict outcomes.

**MM1D3. Students will relate samples to a population.**

- a. Compare summary statistics (mean, median, quartiles, and interquartile range) from one sample data distribution to another sample data distribution in describing center and variability of the data distributions.
- b. Compare the averages of the summary statistics from a large number of samples to the corresponding population parameters.
- c. Understand that a random sample is used to improve the chance of selecting a representative sample.

**MM1D4. Students will explore variability of data by determining the mean absolute deviation (the average of the absolute values of the deviations).**

## COUNTING, PERMUTATIONS, AND COMBINATIONS



### KEY IDEAS

- The ***fundamental counting principle*** states that if one event can occur in  $m$  ways and a second event can occur in  $n$  ways, then the number of ways that both events can occur is  $m \cdot n$  ways. If there is a third event that can occur in  $p$  ways, then the number of ways that all three events can occur is  $m \cdot n \cdot p$  ways. The fundamental counting principle can be extended to any number of events. It can also be used in a situation with or without replacement.

#### Example:

You are asked to type a pass code in order to gain access to an Internet site. You can use the digits 0 to 9 and the letters A to Z. To find the number of different possible pass codes of 1 digit followed by 3 letters when you can use a letter more than one time, multiply  $10 \cdot 26 \cdot 26 \cdot 26 = 175,560$ .

There are 10 different digits for the first position and 26 different letters for the second, third, and fourth positions.

#### Example:

You are asked to type a pass code in order to gain access to an Internet site. You can use the digits 0 to 9 and the letters A to Z. To find the number of different possible pass codes of 1 digit followed by 3 letters when you can NOT use a letter more than once, multiply  $10 \cdot 26 \cdot 25 \cdot 24 = 156,000$ .

There are 10 different digits for the first position, 26 different letters for the second position, 25 different letters for the third position, and 24 different letters for the fourth position.

- Factorial notation*** is indicated by the “!” symbol. The expression  $n!$  is read “ $n$  factorial” and means to multiply  $n$  times  $(n - 1)!$  or  $n! = n(n - 1)!$ . The expression  $0!$  always equals 1.

#### Example:

The expression  $5!$  tells you to start with 5 and multiply it by 4 and 3 and 2 and 1 or  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ .

#### Example:

The expression  $4!$  can be simplified as  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ .

3. A **permutation** is an ordering of a set of objects. When you are concerned with how objects in a set are ordered, you should determine the number of permutations.

4. The number of permutations of  $n$  objects taken  $r$  at a time can be found using

$${}_n P_r = \frac{n!}{(n-r)!}$$

**Example:**

A swim team has 12 swimmers who can swim in a freestyle event. The swim coach will choose 4 of these swimmers for the freestyle relay race. To determine the number of different swimmer orders that are possible for the coach to choose, use the formula for permutations and let  $n = 12$  and  $r = 4$ .

$${}_n P_r = \frac{n!}{(n-r)!} = \frac{12!}{(12-4)!} = \frac{12!}{8!} = 11,880$$

There are 11,880 different possible swimmer orders the coach can choose for the relay race.

5. A **combination** is a selection of objects where the order of the objects is not important.

6. The number of combinations of  $r$  objects taken from a group of  $n$  objects can be found

using  ${}_n C_r = \frac{n!}{r!(n-r)!}$ .

**Example:**

Using the swim team example from Key Idea #4, find the number of possible four-person teams chosen from the eligible swimmers, when the order of the swimmers is NOT important. Use the formula for combinations and let  $n = 12$  and  $r = 4$ .

$$\begin{aligned} {}_n C_r &= \frac{n!}{r!(n-r)!} \\ {}_{12} C_4 &= \frac{12!}{4!(12-4)!} \\ {}_{12} C_4 &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 8!} \\ {}_{12} C_4 &= 495 \end{aligned}$$

There are 495 possible teams of four.

**Example:**

Heather is buying a 2-topping pizza. She has 6 different toppings from which to choose. To find the number of different combinations of 2 toppings that are possible, use

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

$${}_6 C_2 = \frac{6!}{2!(6-2)!}$$

$${}_6 C_2 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)}$$

$${}_6 C_2 = 15$$

There are 15 different possible combinations of 2 toppings.

7. When finding the number of ways that one event or another event can occur, you need to find the number of ways each event can occur and add.

**Example:**

Eric has 3 sandwiches, 2 salads, and 4 drinks. He will choose a sandwich, a salad, and a drink, or he will choose a salad and a drink. Find the number of combinations possible.

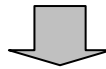
“He will choose a sandwich, a salad, AND a drink”



Multiply

3 sandwiches  $\times$  2 salads  $\times$  4 drinks

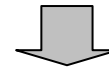
“OR”



ADD

+

“salad AND a drink”



Multiply

2 salads  $\times$  4 drinks

$$(3 \times 2 \times 4) + (2 \times 4) = 24 + 8 = 32 \text{ combinations possible}$$

**REVIEW EXAMPLES**

- 1) Drake can choose from 31 flavors of ice cream. He wants to get a bowl with four scoops of ice cream. Each of the four scoops of ice cream will be a different flavor.

How many different bowls of four scoops of ice cream are possible?

**Solution:**

The event of selecting a flavor of ice cream after the first selection is dependent on the previous event, since each scoop of ice cream will be a different flavor.

- Scoop 1 has 31 options.
- Scoop 2 has  $31 - 1 = 30$  options.
- Scoop 3 has  $30 - 1 = 29$  options.
- Scoop 4 has  $29 - 1 = 28$  options.

Multiply  $31 \times 30 \times 29 \times 28 = 755,160$  different bowls of four scoops of ice cream.

- 2) From a group of 5 nutritionists and 7 nurses, Elyse must select a committee consisting of 2 nutritionists and 3 nurses. In how many ways can she do this if one particular nurse must be on the committee?

**Solution:**

In this scenario, consider that there is more than one event.

First event: choose the nutritionists. Let  $n = 5$  and  $r = 2$ .

$${}^n C_r = \frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!(3)!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{(2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)} = \frac{20}{2} = 10$$

Second event: choose the nurses. The problem states that one particular nurse must be on the committee, so let  $n = 7 - 1 = 6$ , and  $r = 3 - 1 = 2$ .

$${}^n C_r = \frac{6!}{r!(n-r)!} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!(4)!} = \frac{6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{(2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)} = \frac{30}{2} = 15$$

To find the possible committees, multiply the combinations  $10 \times 15 = 150$ .

Elyse has 150 possible ways to form the committee.

**EOCT Practice Items**

- 1) **There are five points in a plane, but no three points are collinear. How many different straight lines that pass through two of the points are possible?**

A. 2  
B. 10  
C. 15  
D. 20

[Key: B]

- 2) **Danny has 3 identical color cubes. Each of the 6 faces on the color cubes is a different color. He also has 2 fair coins.**

**What is the total number of possible outcomes if Danny rolls all three cubes OR flips both coins?**

A. 22  
B. 144  
C. 220  
D. 864

[Key: C]

- 3) **There are 14 students in a mathematics competition. Each student will earn points during the competition. The student with the greatest number of points will be the first place winner, and the student with the second greatest number of points will be the second place winner.**

**How many different ways can the 14 students finish in first place and second place?**

A. 27  
B. 91  
C. 182  
D. 196

[Key: C]

## PROBABILITY



### KEY IDEAS

1. The **probability** of an event is the ratio of the number of outcomes in the event to the total number of outcomes in the sample space. It is the likelihood that an event will occur.

$$P(E) = \frac{\text{number of outcomes in the event}}{\text{total number of outcomes in the sample space}}$$

The probability of an event occurring is represented by a number between 0 and 1. A probability of 0 means that the occurrence of the event is NOT possible. A probability of 1 means that the occurrence of the event is absolutely certain.

2. Two events that have no outcomes in common are called **mutually exclusive** events. They are two events that cannot occur at the same time.
3. Events are **random events** when individual outcomes are uncertain. However, there may be a regular distribution of outcomes in a large number of repetitions. An example of this is flipping a fair coin. If you flip it enough times it will land with heads facing up about 50% of the time and tails facing up about 50% of the time. However, the outcome of a single flip of the coin is uncertain.
4. The **addition rule for mutually exclusive events** is shown by the formula  $P(A \text{ or } B) = P(A) + P(B)$ .

#### Example:

The sample space for rolling a number cube is  $\{1, 2, 3, 4, 5, 6\}$ , with each number representing the top face of the cube. Only one number can be on the top face at a time.

To find the probability of a number cube landing with 2 facing up or landing with 5 facing up (mutually exclusive events), find the probability of the cube landing on 2, and then find the probability of the cube landing on 5. Then find the sum of the two probabilities of the

two events:  $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ .

5. The **addition rule for sets that are NOT mutually exclusive** is shown by the formula  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ .

**Example:**

For a number cube with faces numbered from 1 to 6, the events of “rolling a number cube and having it land with an even number on the top face” and “rolling a number cube and having it land with a number greater than two on the top face” are not mutually exclusive.

Let event  $A$  represent rolling the number cube and having it land with an even number on the top face. Let event  $B$  represent rolling the number cube and having it land with a number greater than two on the top face.

One way to find the probability of  $A$  or  $B$  is to use the formula

$$P(A \text{ or } B) = P(\text{even number}) + P(\text{number greater than two}) - P(\text{even number and number greater than two}) = \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6}.$$

Another way is to examine the sample space  $\{ \quad \}$  and count the number of outcomes that are even numbers or are numbers that are greater than two. Those numbers are 2, 3, 4, 5, and 6. That is 5 out of the 6 possible outcomes. The probability would be  $\frac{5}{6}$ .

6. Two events are ***independent events*** if the outcome of the first event does not affect the probability of the second event.
7. Two events are ***dependent events*** when the outcome of the first event affects the probability of the second event.

**Example:**

Suppose 2 cards are drawn from a standard deck of 52 cards *without replacement*. The probability that both cards are clubs would be  $\frac{13}{52} \cdot \frac{12}{51}$  or  $\frac{1}{17}$ . If a club was drawn first, then there would be only 12 clubs left out of 51 cards left, since the first club was not returned to the deck. The probability of drawing a club on the second draw is different than the probability of drawing a club on the first draw because the events are dependent.

8. ***Conditional probability*** is a type of dependent probability. Given two events, conditional probability is the probability of event  $B$ , given that event  $A$  has occurred. Conditional probability is denoted as  $P(B|A)$  and can be found using  $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$ .



**Example:**

You toss a coin and roll a number cube that has faces numbered from 1 to 6. Find the probability that the number cube will land with an odd number facing up, given that tails is facing up on the coin.

**Solution:**

There are two events:

$$\begin{array}{ll} \text{Event } A: \text{ Tossing a coin} & \text{Event } B: \text{ Rolling a fair number cube} \\ P(A) = \frac{1}{2} \leftarrow \frac{\text{Landed on tails}}{\text{Two outcomes possible}} & P(B) = \frac{3}{6} \leftarrow \frac{3 \text{ odd numbers on a number cube}}{6 \text{ total numbers on a number cube}} \end{array}$$

$$\text{Find } P(B|A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{\frac{1}{2} \cdot \frac{3}{6}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

The probability can also be found by making a list of the sample space. Tails is facing up on the coin, so the sample space is:

$$(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)$$

There are 3 favorable outcomes, (T,1), (T, 3), and (T,5), out of the 6 possible.

$$\text{So, } P(B|A) = \frac{3}{6} = \frac{1}{2}.$$

9. **Expected value** is the sum of the probability of each possible outcome of an event multiplied by the value of each outcome.

Expected value =  $p_1o_1 + p_2o_2 + p_3o_3 + \dots + p_n o_n$ , where  $p$  is the probability of each outcome and  $o$  is the value of each outcome.

This value represents the long-term average of the probability of each outcome weighted by the payoff for that outcome. Keep in mind that the expected value may not be an actual outcome, as it represents an *average* value, which could lie between two actual outcome values.

**Example:**

Suppose you are playing a game with a number cube that has its six faces numbered 1, 1, 1, 1, 2, and 3. The expected value for rolling the cube would be 1.5.

$$\begin{aligned}
 E(\text{number cube}) &= \frac{1}{6}(1) + \frac{1}{6}(1) + \frac{1}{6}(1) + \frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) \\
 &= \frac{1+1+1+1+2+3}{6} \\
 &= \frac{9}{6} \\
 &= 1.5
 \end{aligned}$$

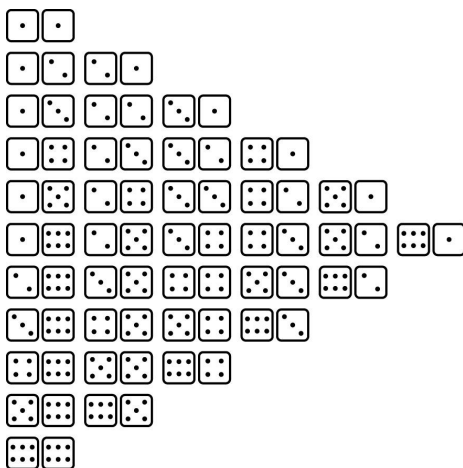
**Note:** The expected value, 1.5, is a value that is impossible to roll, yet represents the average value of a roll of the number cube in the example.

### REVIEW EXAMPLES

- 1) Two number cubes are rolled that each have faces numbered from 1 to 6. What is the probability that the sum of the numbers on the top face of each cube is 4 or 5?

**Solution:**

This diagram shows the sample space for a roll of two cubes organized such that each row shows the possible outcomes for a specific sum.



There are 36 possible outcomes when two cubes are rolled.

There are 3 possible outcomes with a sum of 4.

There are 4 possible outcomes with a sum of 5.

Thus, there are 7 possible outcomes with a sum of 4 or 5 out of 36 possible outcomes, and

the probability is  $\frac{7}{36}$ .

**Check:**

$$P(A \text{ or } B) = P(A) + P(B) = \frac{3}{36} + \frac{4}{36} = \frac{7}{36}$$

- 2) City consultants conducted a survey of 100 people to determine the community interest in constructing a new fire station. The results are shown in this table.

	Males	Females
Supports	27	25
Opposes	18	13
No Opinion	5	12

- Find the probability that a randomly selected survey participant supports the construction of a new fire station or has no opinion.
- Find the probability that a randomly selected survey participant does NOT support the construction of a new fire station.
- Find the probability that a randomly selected survey participant is female or opposes the construction of a new fire station.

**Solution:**

- The number of people who support the fire station is  $27 + 25 = 52$ .  
The number of people who have no opinion is  $5 + 12 = 17$ .

Supporting the fire station or having no opinion are two mutually exclusive events, as a person cannot answer the survey question with both responses at the same time.

$$\begin{aligned}
 P(\text{supporting or no opinion}) &= P(\text{supporting}) + P(\text{no opinion}) \\
 &= \frac{52}{100} + \frac{17}{100} \\
 &= \frac{69}{100}
 \end{aligned}$$

The probability that a randomly selected survey participant supports the construction of a new fire station or has no opinion is  $\frac{69}{100}$  or 69%.

- The probability that a randomly selected survey participant does NOT support the construction of a new fire station is the *complement* of part a.

Recall that a **complement of an event**  $E$  is  $P(\text{not } E) = 1 - P(E)$ . So, to find the probability that a survey participant does NOT support the construction of a new fire station, subtract:

$$100\% - 69\% = 31\% \text{ or } \frac{31}{100}.$$

- c. The probability that a randomly selected survey participant is female is

$\frac{25+13+12}{100} = \frac{50}{100}$ . The probability that a randomly selected survey participant opposes

the construction of a new fire station is  $\frac{18+13}{100} = \frac{31}{100}$ . The probability that a randomly

selected survey participant is female AND against the construction of a new fire station is  $\frac{13}{100}$ . Note that this group of 13 participants has been counted both in the “female” group

and the “opposes the construction of a new fire station” group; this is why it must be subtracted from the total. The probability that a randomly selected survey participant is

female or is against the construction of a new fire station is  $\frac{50}{100} + \frac{31}{100} - \frac{13}{100} = \frac{68}{100}$  or 68%.

- 3) Keira is playing a game at a school carnival. She pays \$1 to play the game. In the game, there are eight identical small doors. Behind six of the doors there is nothing. Behind one of the doors there is a \$1 coupon she can use to play the game again. Behind one of the doors there is a prize worth \$5. The coupon and prize are assigned randomly to doors each time a person plays the game.
- What is the expected value of playing the game one time?
  - What is the net value of playing the game one time?

**Solution:**

- There are eight doors, each of which is an equally likely outcome for the game.
  - The value of each of six of the doors is \$0.
  - The value of one of the doors is \$1.
  - The value of one of the doors is \$5.

The expected value of playing the game one time is \$0.75.

$$\frac{6}{8}(0) + \frac{1}{8}(1) + \frac{1}{8}(5) = \frac{1}{8} + \frac{5}{8} = \frac{6}{8} = \frac{3}{4} \text{ of a dollar, or } \$0.75.$$

- The net value of playing the game is found by taking the expected value of playing the game and subtracting from it the cost of playing the game.

$$\$0.75 - \$1.00 = -\$0.25$$

The net value of playing the game one time is  $-\$0.25$ , or a loss of \$0.25. Note that if Keira plays the game 100 times she can expect to lose a total of  $100 \times \$0.25 = \$25$ .

**EOCT Practice Items**

- 1) A teacher has 9 red crayons, 4 blue crayons, 7 purple crayons, and 5 black crayons in a basket. A student reaches into the basket and randomly selects a crayon. What is the probability that the crayon will be either blue or black?

A.  $\frac{4}{16}$

B.  $\frac{9}{25}$

C.  $\frac{13}{25}$

D.  $\frac{9}{16}$

[Key: B]

- 2) There are 6 red apples, 4 yellow apples, and 2 green apples in a bucket. Maria will choose two apples at random without replacement.

What is the probability that Maria will choose a red apple and then a green apple?

A.  $\frac{5}{121}$

B.  $\frac{6}{121}$

C.  $\frac{1}{11}$

D.  $\frac{1}{12}$

[Key: C]

- 3) This table shows the probability of each possible sum when two cubes with faces numbered 1 through 6 are rolled and the numbers showing on each face are added.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Seth is playing a game in which he gets 10 points when the sum is a perfect square. He gets 5 points if the sum is a prime number. He gets 0 points if the sum is a number that is neither prime nor a perfect square.

What is the expected value, to the nearest 0.1, for one roll of the two number cubes?

- A. 4.0 points
- B. 4.1 points
- C. 4.3 points
- D. 4.6 points

[Key: A]

## SUMMARY STATISTICS



### KEY IDEAS

1. The *measures of central tendency* are mean, median, and mode.
  - The *mean* of a data set can be determined by dividing the sum of the data values by the total number of data values.
  - The *median* of a data set is the middle value of the data set, or the average of the two middle values of a data set with an even number of values when the data is placed in order.
  - The *mode* of a data set is the most frequent value in the data set.

2. The *first quartile* or the *lower quartile*,  $Q_1$ , is the median of the lower half of a data set.

**Example:**

Ray's scores on his first 7 mathematics tests were 70, 85, 78, 90, 84, 82, and 83. To find the first quartile of his scores, put them in order and find the median of the lower half.

70, 78, 82, 83, 84, 85, 90

The first quartile is 78.

3. The *third quartile* or the *upper quartile*,  $Q_3$ , is the median of the upper half of a data set.

**Example:**

To find the third quartile of Ray's scores from the example in Key Idea #2, put the scores in order and find the median of the upper half. The third quartile is 85.

4. The *interquartile range* of a data set is the difference between the quartiles or  $Q_3 - Q_1$ .

**Example:**

To find the interquartile range of Ray's scores from the example in Key Idea #2, subtract the first quartile from the third quartile. The interquartile range of Ray's scores is  $85 - 78 = 7$ .

5. A *population* is a group of people, animals, or objects and a *sample* is part of the population. The general goal of all sampling methods is to obtain a sample that is representative of the target population.

6. One way to obtain a representative sample of the population is to use a random sample. A **random sample** is one where every person in the population from which the sample is drawn has an equal chance of being included.

**Example:**

A teacher wants to randomly select a student. To do this, the names of each student are written on a piece of paper and placed in a paper bag. The teacher then draws a piece of paper from the bag without looking at the name. This method ensures that each student has an equal chance of being selected.

If the teacher selects a student from among those students who raised their hands, this is NOT random. Not every student has an equal chance of being selected.

**REVIEW EXAMPLES**

- 1) John surveys every fifth person leaving a pet supply store. Of those surveyed,  $\frac{3}{4}$  support the city manager's proposition to tear down the old library structures and replace the area with the construction of a new pet park. John plans to write a letter to the editor of the local newspaper about the proposal for the new pet park stating that there is tremendous support from the citizens of the town for constructing a new pet park.
- Can the conclusion John formed be accurately supported?
  - Suggest another plan for obtaining a good sample population.

**Solution:**

- The population that John studied is likely to have a very favorable opinion about pets, as they are shopping in a pet supply store. It is likely that this population of people would be strong supporters for a new pet park and would therefore cause the findings to be biased. To conclude that there is tremendous support from the citizens of the town may not be true because John only sampled the population of people who shopped at a pet store. The population studied was not representative of the people in the entire town.
- To obtain a representative sample from the population of people in the town, select every fiftieth person from the data bank of registered voters. This population would represent the people who will vote on the issue, and likely a good mix of people who are fond of pets, people who are fond of libraries, people who are fond of architecture, or none of the above.



- 2) Warren and Mason each get paid a bonus at the end of each month. This table shows their bonuses for the first five months of the year.

Month	Warren's Bonus	Mason's Bonus
January	\$250	\$250
February	\$250	\$320
March	\$270	\$310
April	\$240	\$300
May	\$260	\$260

- Who had the greatest median bonus? What is the difference in the median of Warren's bonuses and the median of Mason's bonuses?
- What is the difference in the interquartile range for Warren's bonus and Mason's bonus?

**Solution:**

- Warren's median bonus is \$250. Put Warren's bonuses in order and find the middle one.

\$240, \$250, **\$250**, \$260, \$270

Mason's median bonus is \$300. Put Mason's bonuses in order and find the middle one.

\$250, \$260, **\$300**, \$310, \$320

Mason had the greatest median bonus. The difference in the median of the bonuses is

$$300 - 250 = 50 \text{ or } \$50.$$

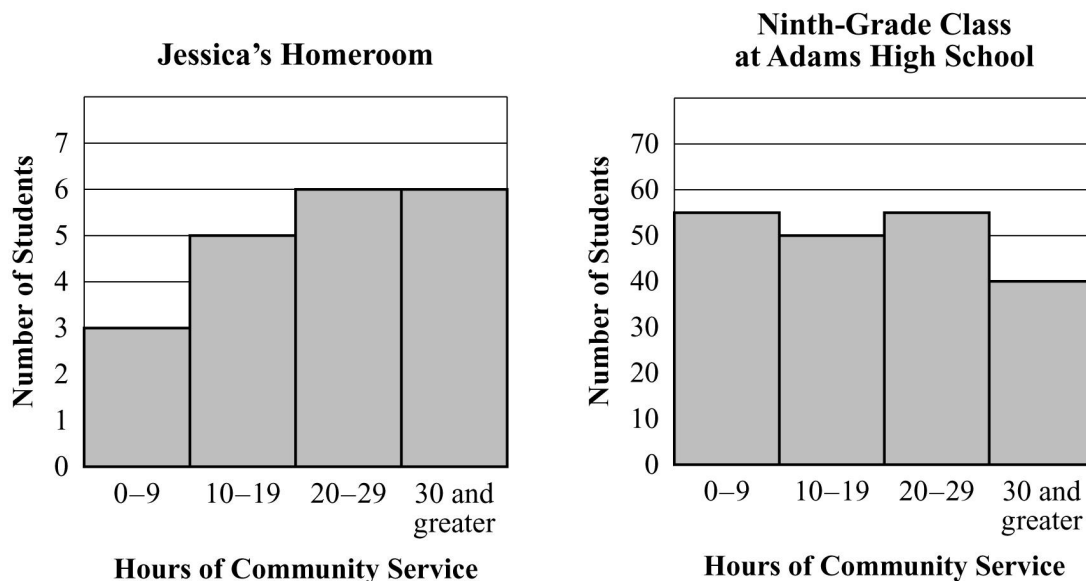
- The lower quartile for Warren is \$245 and the upper quartile is \$265. The interquartile range is the difference in these or \$20.

The lower quartile for Mason is \$255 and the upper quartile is \$315. The interquartile range is the difference in these or \$60.

The difference in Warren's interquartile range and Mason's interquartile range is

$$60 - 20 = 40 \text{ or } \$40.$$

- 3) Jessica is a student at Adams High School. These histograms give information about the number of hours of community service completed by each of the students in Jessica's homeroom and by each of the students in the ninth-grade class at her school.



- Compare the lower quartiles of the data in the histograms.
- Compare the upper quartiles of the data in the histograms.
- Compare the medians of the data in the histograms.

**Solution:**

- You can add the number of students in each bar to find there are 20 students in Jessica's homeroom. The lower quartile is the median of the first half of the data. That would be found within the second bar, which represents 10-19 hours.

You can add the number of students in each bar to find out there are 200 students in the ninth-grade class. The lower quartile for this group is found within the first bar, which represents 0-9 hours.

The lower quartile of the number of community service hours completed by each student in Jessica's homeroom is greater than the lower quartile of community service hours completed by each student in the ninth-grade class.

- The upper quartile is the median of the second half of the data. For Jessica's homeroom, that would be found within the fourth bar, which represents 30 and greater hours.

For the ninth-grade class, the upper quartile is found within the third bar, which represents 20-29 hours.

The upper quartile of the number of community service hours completed by each student in Jessica's homeroom is greater than the upper quartile of community service hours completed by each student in the ninth-grade class.

- c. The median is the middle number in a data set that is written in order from least to greatest or greatest to least. The median for Jessica's homeroom is found within the third bar, which represents 20-29 hours.

The median for the ninth-grade class is found within the second bar, which represents 10-19 hours.

The median of the number of community service hours completed by each student in Jessica's homeroom is greater than the median of community service hours completed by each student in the ninth-grade class.

**EOCT Practice Items**

- 1) This table shows the average high temperature, in °F, recorded in Atlanta, GA, and Austin, TX, over a six-day period.

Day	1	2	3	4	5	6
Temperature in °F in Atlanta, GA	85	88	83	79	81	85
Temperature in °F in Austin, TX	80	86	82	80	93	89

Which conclusion can be drawn from the data?

- A. The median temperature over the six-day period was the same for both cities, but the interquartile range is greater for Austin than Atlanta.
- B. The mean temperature of Atlanta was higher than the mean temperature of Austin, and the interquartile range is greater for Atlanta than Austin.
- C. The mean temperature of Austin was higher than the mean temperature of Atlanta, but the median temperature of Austin was lower.
- D. The mean and median temperatures of Atlanta were higher than the mean and median temperatures of Austin.

[Key: A]

- 2) A school was having a canned food drive for a local food bank. A teacher determined the median number of cans collected per class and the interquartile ranges of the number of cans collected per class for the juniors and for the seniors.

- The juniors collected a median number of cans per class of 35, and the interquartile range was 10.
- The seniors collected a median number of cans per class of 40, and the interquartile range was 8.
- Both the juniors and the seniors had the same third quartile number of cans collected.

Which range includes only the numbers that could be the third quartile number of cans collected for both classes?

- A. 25 to 45
- B. 25 to 48
- C. 32 to 48
- D. 40 to 45

[Key: D]

## MEAN ABSOLUTE DEVIATION



### KEY IDEAS

1. The **mean absolute deviation** is a measure of spread (or variability). It is the mean amount by which the values in a data set differ, or vary, from the mean.
2. The mean absolute deviation of a set of data can be determined by following these steps:
  - Calculate the mean of the data set.
  - Find the absolute values of the differences of each data point and the mean.
  - Find the sum of the absolute values.
  - Divide the sum by the total number of values in the data set.

#### Example:

The mean of the data set, 6, 6, 9, 4, 5 is 6. To determine the mean absolute deviation, find the absolute values of the differences of each data point and 6 (the mean).

$$|6 - 6| = 0; |6 - 6| = 0; |9 - 6| = 3; |4 - 6| = 2; |5 - 6| = 1$$

$$\text{Find the sum: } 0 + 0 + 3 + 2 + 1 = 6$$

$$\text{Divide 6 by the total number of data values: } 6 \div 5 = 1.2$$

The mean absolute deviation is 1.2.

### REVIEW EXAMPLES

- 1) What is the mean absolute deviation of the following data set?

25, 57, 44, 34

#### Solution:

$$\text{First find the mean of the data set: } \frac{25 + 57 + 44 + 34}{4} = 40$$

Then find the absolute values of the difference between each data point and 40.

$$|40 - 25| + |40 - 57| + |40 - 44| + |40 - 34| = 42$$

$$\text{Divide the sum of 42 by the number of data points: } 42 \div 4 = 10.5$$

The mean absolute deviation of the set is 10.5.

2) Emma and Sara play 5 games on a handheld video game and record their scores in this table.

Game Score		
Game	Emma	Sara
1	30	15
2	20	25
3	10	20
4	20	25
5	20	15

Which girl had the greater mean deviation for her scores?

**Solution:**

Emma's mean score is  $\frac{30 + 20 + 10 + 20 + 20}{5} = 20$

Find the absolute values of the differences between each data point and 20.

$$|30 - 20| = 10; |20 - 20| = 0; |10 - 20| = 10; |20 - 20| = 0; |20 - 20| = 0$$

Find the sum:  $10 + 0 + 10 + 0 + 0 = 20$

Divide the sum of 20 by the total number of data values:  $20 \div 5 = 4$

Sara's mean score is  $\frac{15 + 25 + 20 + 25 + 15}{5} = 20$

Find the absolute values of the differences of each data point and 20.

$$|15 - 20| = 5; |25 - 20| = 5; |20 - 20| = 0; |25 - 20| = 5; |15 - 20| = 5$$

Find the sum:  $5 + 5 + 0 + 5 + 5 = 20$

Divide the sum of 20 by the total number of data values:  $20 \div 5 = 4$

The absolute mean deviation is 4 for both Emma and Sara. Neither one had a greater mean absolute deviation.

**EOCT Practice Items**

- 1) This table shows the scores of four students on their first four mathematics quizzes.

Student	Quiz Scores	Mean Score
Anna	80, 90, 75, 75	80
Jim	100, 70, 80, 70	80
Stacy	92, 78, 68, 82	80
Ted	71, 84, 84, 81	80

Which student had the least mean absolute deviation on the quiz scores?

- A. Anna
- B. Jim
- C. Stacy
- D. Ted

[Key: D]

- 2) The heights, in inches, of five girls in an exercise class were 66, 64, 68, 70, and 65. A sixth girl joined the class. The mean height of the six girls in the class was 66 inches and the mean absolute deviation was 2 inches. What was the height of the sixth girl who joined the class?

- A. 63 inches
- B. 64 inches
- C. 65 inches
- D. 66 inches

[Key: A]

## **Unit 2: Algebra Investigations**

The focus of this unit is the development of students' abilities to read and write the symbolically intensive language of algebra. Students develop skills in adding, subtracting, multiplying, and dividing elementary polynomial, rational, and radical expressions. By using algebraic expressions to represent quantities in context, students understand algebraic rules as general statements about operations on real numbers. Work with products and the zero factor property are introduced.

### **KEY STANDARDS**

#### **MM1A2. Students will simplify and operate with radical expressions, polynomials, and rational expressions.**

- a. Simplify algebraic and numeric expressions involving square root.
- b. Perform operations with square roots.
- c. Add, subtract, multiply, and divide polynomials.
- d. Expand binomials using the Binomial Theorem.
- e. Add, subtract, multiply, and divide rational expressions.
- f. Factor expressions by greatest common factor, grouping, trial and error, and special products limited to the formulas below.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)(x - y) = x^2 - y^2$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

- g. Use area and volume models for polynomial arithmetic.

#### **MM1A3. Students will solve simple equations.**

- a. Solve quadratic equations in the form  $ax^2 + bx + c = 0$ , where  $a = 1$ , by using factorization and finding square roots where applicable.



## ALGEBRAIC EXPRESSIONS



### KEY IDEAS

1. An *algebraic expression* is an expression that contains at least one variable.
2. A *polynomial* is an algebraic expression that contains one or more monomials separated by + or – signs. Each monomial is a *term* of the polynomial.
3. To add or subtract polynomials, combine like terms. Recall that *like terms* have the same variable(s) with the same exponent(s).

**Example:**

$$\begin{aligned}
 &(7x^4 - 3x^2) + (9x^4 - 2x^2 + 6) \\
 &= (7x^4 + 9x^4) + (-3x^2 - 2x^2) + 6 && \text{One way is to use the Commutative and Associative Properties} \\
 & && \text{to reorder and regroup the terms.} \\
 &= 16x^4 - 5x^2 + 6 && \text{Combine like terms.}
 \end{aligned}$$

4. Use the *distributive property* to multiply polynomials.

**Example:**

$$\begin{aligned}
 4x(-3x^2 + 4x - 2) &= 4x(-3x^2) + (4x)(4x) - (4x)(2) && \text{Distribute } 4x \text{ to each term.} \\
 &= -12x^3 + 16x^2 - 8x && \text{Multiply coefficients and add exponents.}
 \end{aligned}$$

**Example:**

$$\begin{aligned}
 &(x - 4)(x + 7) \\
 &= x^2 + 7x - 4x - 28 && \text{Distribute } x \text{ in the first parentheses to each term} \\
 & && \text{in the second parentheses. Then distribute } -4 \\
 & && \text{to each term in the second parentheses.} \\
 &= x^2 + 3x - 28 && \text{Combine like terms.}
 \end{aligned}$$

5. These are the formulas for six special products that always result in a pattern.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)(x - y) = x^2 - y^2$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

6. To **factor** a polynomial means to write the polynomial as the product of factors. In the list of special products, the polynomial is the expression to the right of the equal sign and the factored form is the expression to the left of the equal sign.

7. To factor an expression, one or more of the following factoring methods may have to be performed. A variety of factoring methods can be used to factor an expression.

- Factoring out a greatest common factor.

$$2x^2 + 6x + 10 = 2(x^2 + 3x + 5)$$

- Factoring by grouping.

$$8 + xy + 4y + 2x = (2x + 8) + (xy + 4y) = 2(x + 4) + y(x + 4) = (2 + y)(x + 4)$$

- Factoring a trinomial into two binomials using trial and error.

$$x^2 + 2x - 15 = (x + \boxed{?})(x + \boxed{?}) = (x + (-3))(x + 5)$$

- Factoring using patterns of special products.

$$\text{Difference of two squares: } x^2 - y^2 = (x + y)(x - y)$$

$$\text{Square of a binomial: } (x + y)^2 = x^2 + 2xy + y^2 \quad \text{or} \quad (x - y)^2 = x^2 - 2xy + y^2$$

$$\text{Cube of a binomial: } (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \quad \text{or} \quad (x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

8. The **Zero Product Property** states that if  $a$  and  $b$  are real numbers such that the product  $ab$  equals 0, then  $a = 0$  or  $b = 0$ .

9. The **Binomial Theorem** can be used to determine powers of binomials. This theorem is helpful in learning how to expand binomials, such as  $(x + y)^n$ , especially for large values of  $n$ . Without this theorem,  $(x + y)$  would have to be multiplied by itself  $n$  times. For the EOCT, you will need to know how to expand binomials with  $n$ , at most, equaling 3.

### The Binomial Theorem

Let  $n$  be a positive integer.

$$(x + y)^n = x^n + \frac{n}{1}x^{n-1}y^1 + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \dots y^n$$

- There are  $n + 1$  terms.
- The first term is  $x^n$  and the last term is  $y^n$ .
- The sum of the exponents in each term is  $n$ .
- The exponent of  $x$  in any term is one less than the exponent of  $x$  in the preceding term.
- The exponent of  $y$  in any term is one more than the exponent of  $y$  in the preceding term.
- The coefficient of any term in an expansion is  $\frac{n!}{(\text{exponent of } x)!(\text{exponent of } y)!}$ .

The Binomial Theorem can also be written in **factorial notation**.

$$(x + y)^n = x^n + \frac{n!}{(n-1)!1!}x^{n-1}y + \frac{n!}{(n-2)!2!}x^{n-2}y^2 + \dots y^n$$

#### Example:

Expand  $(5x - y)^3$ .

#### Solution:

In the binomial  $(5x - y)^3$ ,  $n = 3$ . So, the expansion will have  $n + 1$ , or 4, terms.

$$(5x - y)^3 = [5x + (-y)]^3 \quad \text{Rewrite expression in the form } (x + y)^n.$$

Follow the Binomial Theorem.

$$\begin{aligned} &= (5x)^3 + \frac{3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 1}(5x)^2(-y)^1 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 1}(5x)^1(-y)^2 + (-y)^3 \\ &= 125x^3 - 75x^2y + 15xy^2 - y^3 \end{aligned}$$

Notice that there are 4 terms, the variable for the first term is  $x^3$ , the variable for the last term is  $y^3$ , and the sum of the exponents in each term is 3.

10. **Equivalent expressions** give the same numerical value output for every number in the domain for the expressions.

#### Example:

Consider the sequence: 3, 6, 9, 12, 15, ... There are different ways to look at the pattern of the sequence.

*One way* is to say each number in the sequence is 3 times the number of its step. The fourth step would be 3 times 4 or 12. This can be written as the algebraic expression  $3n$ , where  $n$  stands for the number of the step.

*Another way* is to say each number in the sequence is 3 added to 3 times the number of the step before it. The fourth step would be 3 added to 3 times 3 (the number of the step before it) or 12. This can be written as the algebraic expression  $3 + 3(n - 1)$ , where  $n$  stands for the number of the step.

The expressions  $3n$  and  $3 + 3(n - 1)$  are equivalent expressions because they represent the same sequence. They give the same numerical value output for every number in the domain for the expressions.

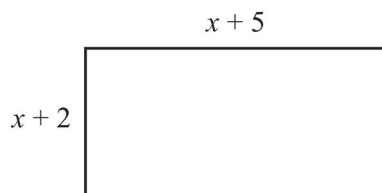
11. No conclusion can be drawn when substituting only one numerical value into two different algebraic expressions and getting the same result for both expressions.
12. An important conclusion can be drawn when substituting one numerical value into two different algebraic expressions and getting *different* results for the two expressions. The expressions are **not** equivalent. It takes only one counterexample to conclude that they are not equivalent.
13. For algebraic expressions that define *linear functions*, it is sufficient to check that two numerical values substituted into the expressions give the same result. This is because the graph of the function is a line and two points are enough to determine a line.

**REVIEW EXAMPLES**

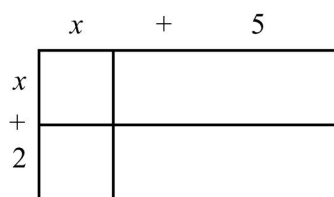
1) Use a geometric figure to find the product of  $(x+5)(x+2)$ .

**Solution:**

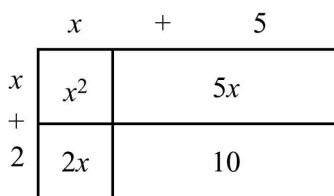
Model the product using a rectangle. First draw a rectangle. Label the length  $(x+5)$ . Label the width  $(x+2)$ .



The area of this rectangle is the product of the length and the width:  $(x+5)(x+2)$ . Now divide the rectangle into four regions, as shown below.



Find the area of each region.

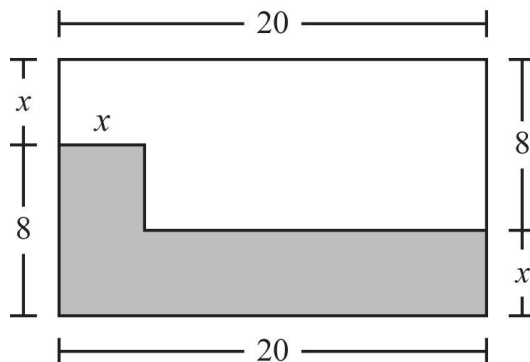


Then find the sum of the areas to find the total area of the rectangle.

$$\begin{aligned} A &= 10 + 2x + 5x + x^2 \\ &= x^2 + 7x + 10 \end{aligned}$$

So,  $(x+5)(x+2) = x^2 + 7x + 10$ .

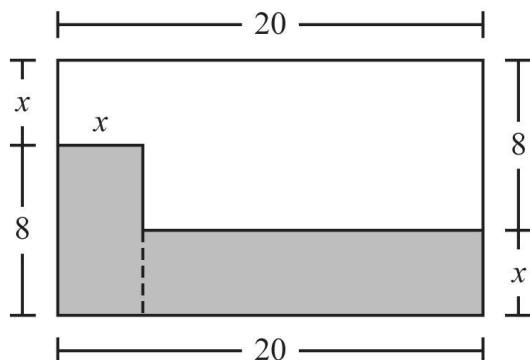
2) This rectangle shows the floor plan of an office.



The shaded part of the plan is an area that is getting new tile. Write an algebraic expression that represents the area of the office that is getting new tile.

**Solution:**

It is helpful to divide the figure into rectangles with dimensions that will make it easy to calculate the area. There is more than one way to do this, but we will use this division of the rectangle.



Now we have one shaded rectangle that is 8 units long and  $x$  units wide and another shaded rectangle that is  $20 - x$  units long and  $x$  units wide.

$$\begin{aligned}
 &(8)(x) + (20 - x)(x) && \text{Add the products of the lengths and widths} \\
 & && \text{of the rectangles.} \\
 &= 8x + 20x - x^2 && \text{Multiply.} \\
 &= 28x - x^2 && \text{Combine like terms.}
 \end{aligned}$$

The area, in square units, is represented by  $28x - x^2$ .

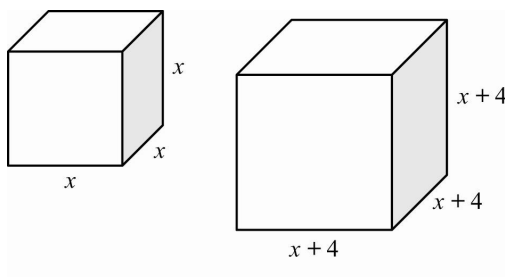
**EOCT Practice Items**

- 1) A train travels at a rate of  $(4x + 5)$  miles per hour. How many miles can it travel at that rate in  $(x - 1)$  hours?

- A.  $3x - 4$  miles  
 B.  $5x - 4$  miles  
 C.  $4x^2 + x - 5$  miles  
 D.  $4x^2 - 9x - 5$  miles

[Key: C]

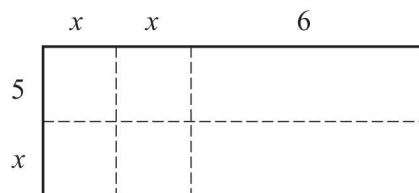
- 2) Taylor and Susan each have a box that is in the shape of a cube. The edges of Taylor's box are each  $x$  cm in length. The edges of Susan's box are 4 cm longer than on Taylor's cube. What binomial expansion represents the volume of Susan's box?



- A.  $(x^2 + 8x + 16)\text{cm}^3$   
 B.  $(x^3 + 12x^2 + 48x + 64)\text{cm}^3$   
 C.  $(x^3 + 768x^2 + 192x + 4)\text{cm}^3$   
 D.  $(x^4 + 12x^3 + 96x^2 + 256x + 256)\text{cm}^3$

[Key: B]

- 3) What is the product of the expression represented by the model below?



- A.  $3x + 11$   
 B.  $x^3 + 30$   
 C.  $2x^2 + 10x + 36$   
 D.  $2x^2 + 16x + 30$

[Key: D]

## RATIONAL EXPRESSIONS



### KEY IDEAS

1. A **rational expression** is the ratio of two polynomials,  $\frac{P_1}{P_2}$ , where  $P_2 \neq 0$ .
2. When the numerator and denominator in the rational expression have no common factors other than 1, then the expression is in **simplest form**.
3. To multiply rational expressions:
  - factor each numerator and denominator
  - divide out common factors
  - multiply the numerators and multiply the denominators
  - simplify the product if it is not in simplest form

#### Example:

$$\begin{aligned} & \frac{4x}{8x+2} \cdot \frac{2x+2}{4x+4} \\ &= \frac{(2)(2)(x)}{(2)(4x+1)} \cdot \frac{(2)(x+1)}{(2)(2)(x+1)} \quad \text{Factor.} \\ &= \frac{x}{4x+1} \cdot \frac{1}{1} \quad \text{Divide out common factors.} \\ &= \frac{x}{4x+1} \quad \text{Multiply the numerators. Multiply the denominators.} \end{aligned}$$

4. To divide rational expressions:
  - multiply the first expression by the reciprocal of the second expression
  - follow the rules for multiplying rational expressions

#### Example:

$$\begin{aligned} & \frac{5x}{15x^2} \div \frac{25x}{x^2} \\ &= \frac{5x}{15x^2} \cdot \frac{x^2}{25x} \quad \text{Multiply by the reciprocal.} \\ &= \frac{(5)(x)}{(3)(5)(x)(x)} \cdot \frac{(x)(x)}{(5)(5)(x)} \quad \text{Factor.} \\ &= \frac{1}{3} \cdot \frac{1}{25} = \frac{1}{75} \quad \text{Divide out common factors and multiply.} \end{aligned}$$



5. To add or subtract rational expressions with like denominators, combine like terms in the numerator and keep the denominator the same.

**Example:**

$$\begin{aligned} \frac{4x}{9x^2} - \frac{2x-3}{9x^2} &= \frac{4x}{9x^2} - \frac{(2x-3)}{9x^2} && \text{Rewrite using parentheses around the numerator of the second fraction.} \\ &= \frac{4x-1(2x-3)}{9x^2} && \text{Rewrite with } -1 \text{ multiplied by the parentheses.} \\ &= \frac{4x-2x+3}{9x^2} && \text{Distribute } -1. \\ &= \frac{2x+3}{9x^2} && \text{Combine like terms. Keep the denominator the same.} \end{aligned}$$

6. To add or subtract rational expressions with unlike denominators:
- find the least common denominator (LCD) of each expression by writing the prime factorization of each denominator
  - use the LCD to write equivalent expressions
  - follow the rules for adding and subtracting rational expressions with like denominators

**Example:**

$$\begin{aligned} \frac{6}{3y} + \frac{1}{3y+6} &= \frac{6}{(3)(y)} + \frac{1}{(3)(y+2)} && \text{Factor. The LCD is } (3)(y)(y+2). \\ &= \frac{6}{(3)(y)} \left( \frac{y+2}{y+2} \right) + \frac{1}{(3)(y+2)} \left( \frac{y}{y} \right) && \text{Build each fraction by multiplying by a fraction} \\ & && \text{equal to 1 that will make the denominators common.} \\ &= \frac{6y+12}{3y(y+2)} + \frac{y}{3y(y+2)} \\ &= \frac{7y+12}{3y(y+2)} && \text{Combine like terms in the numerators and} \\ & && \text{keep the denominator the same.} \end{aligned}$$

Be sure answer is in simplest form.

**REVIEW EXAMPLES**

- 1) Greg hiked 10 miles from a ranger station to a campground on Monday. On Tuesday he hiked back to the ranger station. The campground was uphill from the ranger station, so his average rate of speed to the campground was 2 miles per hour slower than it was to the ranger station.

Let  $r$  represent Greg's average rate of speed to the ranger station. Write an expression that represents the total time, in hours, that Greg hiked.

**Solution:**

First write an expression that represents the time it took Greg to hike to the campground and an expression that represents the time it took him to hike to the ranger station. Add the two together and simplify. Remember that distance = rate  $\cdot$  time, so time =  $\frac{\text{distance}}{\text{rate}}$ .

The time to the campground was  $\frac{10}{r-2}$ , and the time to the ranger station was  $\frac{10}{r}$ .

$$\frac{10}{r-2} + \frac{10}{r} =$$

The LCD is  $r(r-2)$ .

$$\frac{10}{(r-2)}\left(\frac{r}{r}\right) + \frac{10}{r}\left(\frac{r-2}{r-2}\right) =$$

Build each fraction by multiplying by

a fraction equal to 1 that will make the denominators common.

$$\frac{10r}{r(r-2)} + \frac{10r-20}{r(r-2)} =$$

Simplify the numerators.

$$\frac{20r-20}{r(r-2)}$$

Combine like terms in the numerators

and keep the same denominator.

Be sure answer is in simplest terms.

The total time Greg hiked can be represented by  $\frac{20r-20}{r(r-2)}$ .

2) Shannon had  $x$  cookies. David had 5 more cookies than Shannon.

- Shannon ate half of her cookies.
- David ate one-third of his cookies.

Write an algebraic expression in simplest form that could represent the total number of cookies that David and Shannon ate.

**Solution:**

Since  $x$  represents the number of cookies Shannon had, then  $x + 5$  can represent the number of cookies David had.

Now represent the number of cookies that each one ate by multiplying each expression by the fractional part of the cookies eaten.

The number of cookies that Shannon ate was  $\frac{1}{2}x$  or  $\frac{x}{2}$ .

The number of cookies that David ate was  $\frac{1}{3}(x + 5)$  or  $\frac{x + 5}{3}$ .

To get the total number of cookies eaten, add the two fractional parts together.

$$\frac{x}{2} + \frac{x + 5}{3}$$

$$= \frac{x}{2} \left( \frac{3}{3} \right) + \frac{x + 5}{3} \left( \frac{2}{2} \right)$$

Find the common denominator.

Build the fractions by multiplying each fraction by a fraction that equals 1 that will make the denominator common.

$$= \frac{3x}{6} + \frac{2x + 10}{6}$$

$$= \frac{5x + 10}{6}$$

Multiply.

Combine numerators and keep denominator the same.

Be sure the answer is in simplest form. This is in simplest form because the numerator and the denominator do not have any factors in common.

**EOCT Practice Items**

1) Which expression represents the area of a rectangle given that the length is  $\frac{4d-12}{d^2-9}$  and the width is  $3d+9$ ?

A. 7

B. 12

C.  $\frac{4}{3(d+3)^2}$

D.  $\frac{7d-3}{d^2-9}$

[Key: B]

2) Which expression is equivalent to  $3 \div \frac{9}{27-x}$ ?

A.  $\frac{27-x}{3}$

B.  $\frac{9-x}{9}$

C.  $\frac{3}{9-x}$

D.  $\frac{3}{27-x}$

[Key: A]

## RADICAL EXPRESSIONS



### KEY IDEAS

1. The opposite process of squaring a number is finding its **square root**. The square root of any number,  $x$ , is the number that when multiplied by itself equals  $x$ . Recall that the symbol  $\sqrt{\quad}$  is called a **radical symbol**, and the number under the radical is called a **radicand**.
2. **Product Property of Radicals:** for all real numbers  $x$  and  $y$ , where  $x > 0$  and  $y > 0$ ,  $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ .

**Example:**

$$\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$$

**Example:**

$$\begin{aligned} \sqrt{x^4 y^3} &= \sqrt{x^4} \cdot \sqrt{y^2} \cdot \sqrt{y} && \text{Rewrite as the product of the square roots of the factors.} \\ &= x^2 y \sqrt{y} && \text{Simplify.} \end{aligned}$$

3. **Quotient Property of Radicals:** for all real numbers  $x$  and  $y$ , where  $x > 0$  and  $y > 0$ ,  $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ .

**Example:**

$$\sqrt{\frac{100}{16}} = \frac{\sqrt{100}}{\sqrt{16}} = \frac{10}{4} = \frac{5}{2}$$

**Example:**

$$\sqrt{\frac{x^5}{y^6}} = \frac{\sqrt{x^5}}{\sqrt{y^6}}$$

Rewrite as a quotient of the square roots.

$$= \frac{\sqrt{x^4} \cdot \sqrt{x}}{\sqrt{y^6}}$$

Rewrite the numerator as the product of the square roots of the factors.

$$= \frac{x^2 \sqrt{x}}{y^3}$$

Simplify.

4. To add or subtract radicals, the expressions must be like radicals, that is they must have the exact same radicand. Add or subtract the coefficients and keep the radicand the same.

**Example:**

$$2\sqrt{ab} + 5\sqrt{ab} = 7\sqrt{ab}$$

**Example:**

$$3\sqrt{4x^3} + 7x\sqrt{x}$$

At first it does not look like these can be combined.

$$= 3\sqrt{4} \cdot \sqrt{x^2} \cdot \sqrt{x} + 7x\sqrt{x}$$

Factor where possible.

$$= 6x\sqrt{x} + 7x\sqrt{x}$$

Simplify. Now there are like terms.

$$= 13x\sqrt{x}$$

Combine the like terms.

5. The answer to a problem that involves radicals may be in radical form. If the problem involves finding a length or distance, a calculator may be needed to find the approximate value of the expression.

**Example:**

Cal was making a rectangular flag that was 10 inches by 16 inches and a rectangular flag that was 12 inches by 18 inches. He wanted to buy enough ribbon to attach a strip down the diagonal of each flag. What is the minimum whole number of inches of ribbon that Cal should buy?

**Solution:**

First find the diagonal of each flag by using the Pythagorean theorem.

**First Flag**

$$a^2 + b^2 = c^2$$

$$10^2 + 16^2 = c^2$$

$$100 + 256 = c^2$$

$$c^2 = 356$$

$$c = \sqrt{356}$$

**Second Flag**

$$a^2 + b^2 = c^2$$

$$12^2 + 18^2 = c^2$$

$$144 + 324 = c^2$$

$$c^2 = 468$$

$$c = \sqrt{468}$$

The sum of the lengths of the two diagonals is  $\sqrt{356}$  inches +  $\sqrt{468}$  inches. Since we want to find out how many inches of ribbon Cal should buy, we need to use a calculator to get an approximate value for each square root.

$$\sqrt{356} \approx 18.9 \text{ inches}$$

$$\sqrt{468} \approx 21.6 \text{ inches}$$

Add and round up to the nearest whole inch of ribbon:  $18.9 + 21.6 = 40.5$ . Cal should buy 41 inches of ribbon.

**REVIEW EXAMPLES**

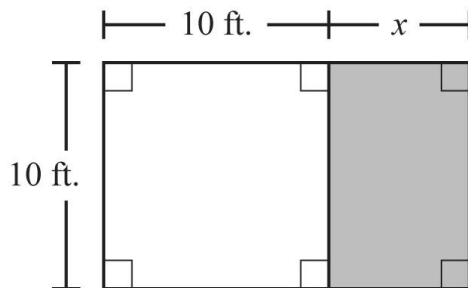
- 1) A square patio has an area of  $98x^2$  square feet. Write an expression that represents the length of one side of the patio.

**Solution:**

To find the length of a side of a square where the area is given, take the square root of the area.

$$\begin{aligned} \sqrt{98x^2} &= \sqrt{49} \cdot \sqrt{2} \cdot \sqrt{x^2} && \text{Rewrite as the product of the square roots of the factors.} \\ &= 7x\sqrt{2} && \text{Simplify.} \end{aligned}$$

- 2) Katie set up a face painting booth at the school carnival. She started with a space in the shape of a square with a side length of 10 feet. She increased the length of the booth by  $x$  feet. The white area in this diagram shows the original shape of the booth, and the shaded area shows the extra part Katie added to it.



Katie plans to hang streamers diagonally across her booth. Write an expression that represents the length of the diagonal of the booth.

**Solution:**

To find the diagonal, in feet, of the booth, use the Pythagorean theorem using  $10 + x$  for the length of the booth and 10 for the width.

$$a^2 + b^2 = c^2$$

$$(x + 10)^2 + 10^2 = c^2$$

Rewrite the Pythagorean theorem using the representations of the length and the width of the booth.

$$x^2 + 20x + 100 + 100 = c^2$$

Square the two dimensions.

$$c^2 = x^2 + 20x + 200$$

Switch the order of the equation and simplify.

$$c = \sqrt{x^2 + 20x + 200}$$

Solve by taking the square root of each side.

Note that the expression under the radical will not factor and that there is no way to take the square root of this expression. This is the answer in simplest form.



**EOCT Practice Items**

1) What value of  $x$  makes the equation  $\sqrt{x} + 3 = 7$  true?

- A. 4
- B. 10
- C. 16
- D. 21

[Key: C]

2) A right isosceles triangle has a hypotenuse with a length represented by  $4y$ . Which expression represents the length of one of the legs of the triangle?

- A.  $2y$
- B.  $4y$
- C.  $y\sqrt{2}$
- D.  $2y\sqrt{2}$

[Key: D]

## Unit 3: Function Families

This unit explores properties of basic quadratic, cubic, absolute value, square root, and rational functions and new language and notation for talking about functions. The discussion of sequences focuses primarily on sequences of numbers and often-used geometric figures and diagrams as illustrations and contexts for investigating various number sequences.

### KEY STANDARDS

**MM1A1. Students will explore and interpret the characteristics of functions, using graphs, tables, and simple algebraic techniques.**

- a. Represent functions using function notation.
- b. Graph the basic functions  $f(x) = x^n$  where  $n = 1$  to  $3$ ,  $f(x) = \sqrt{x}$ ,  $f(x) = |x|$ , and  $f(x) = \frac{1}{x}$ .
- c. Graph transformations of basic functions including vertical shifts, stretches, and shrinks, as well as reflections across the  $x$ - and  $y$ -axes.
- d. Investigate and explain the characteristics of a function: domain, range, zeros, intercepts, intervals of increase and decrease, maximum and minimum values, and end behavior.
- e. Relate to a given context the characteristics of a function, and use graphs and tables to investigate its behavior.
- f. Recognize sequences as functions with domains that are whole numbers.
- g. Explore rates of change, comparing constant rates of change (i.e., slope) versus variable rates of change. Compare rates of change of linear, quadratic, square root, and other function families.

# FUNCTIONS

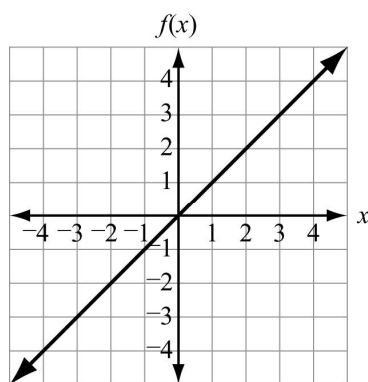


## KEY IDEAS

- Every function can be classified as a member of a “family.” The “parent” of a function family is the most basic representation of the family. These are graphs of some basic **parent functions**.

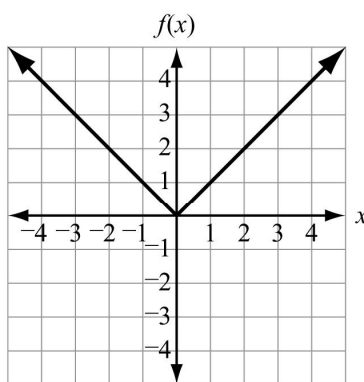
**Linear Function**

$$f(x) = x$$



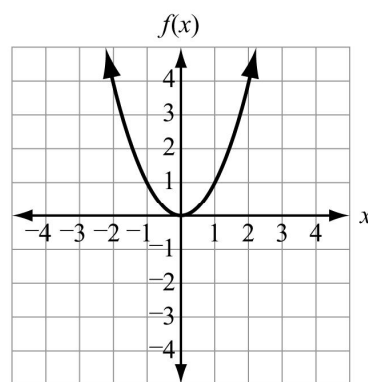
**Absolute Value Function**

$$f(x) = |x|$$



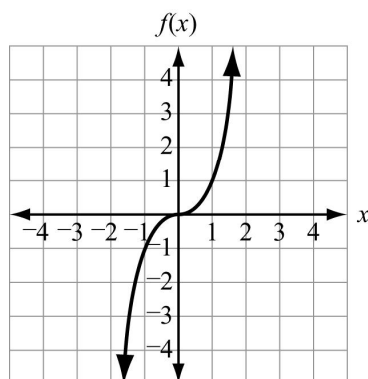
**Quadratic Function**

$$f(x) = x^2$$



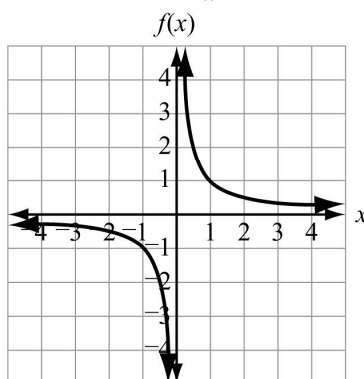
**Cubic Function**

$$f(x) = x^3$$



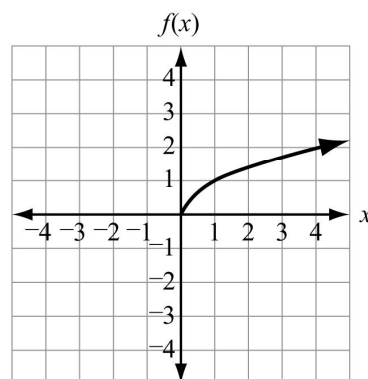
**Reciprocal Function**

$$f(x) = \frac{1}{x}$$



**Square Root Function**

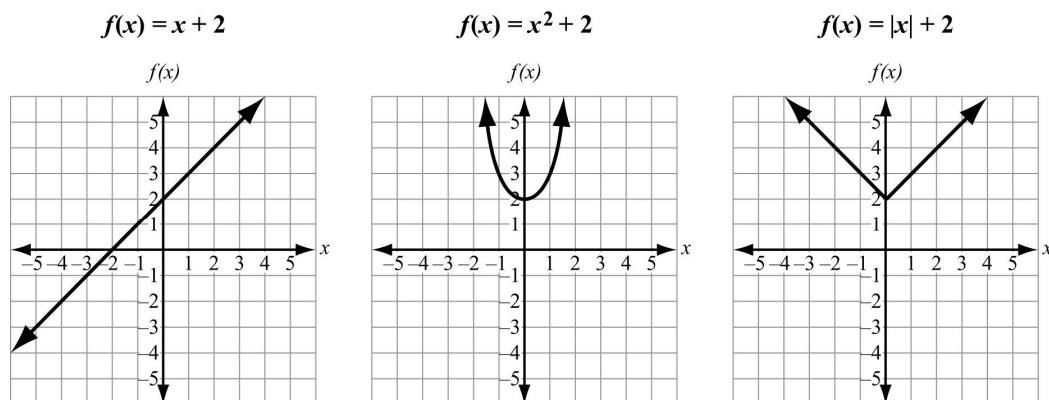
$$f(x) = \sqrt{x}$$



Parent functions can be transformed into many new functions. By studying the characteristics of a family of functions, you can determine how a parent graph is transformed.

**Example:**

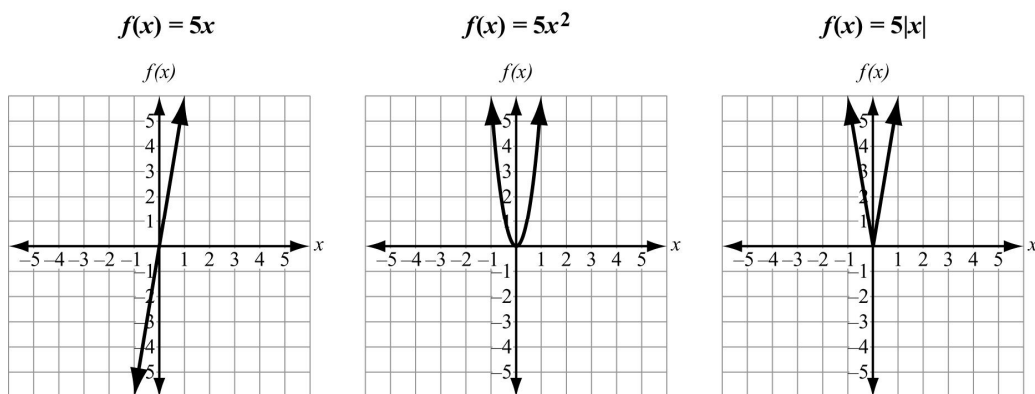
Graph  $f(x) = x + 2$ ,  $f(x) = x^2 + 2$ , and  $f(x) = |x| + 2$ . Describe how adding 2 to each parent function value affects the corresponding parent graph.

**Solution:**

When 2 is added to each parent function, the corresponding parent graphs are vertically translated up 2 units.

**Example:**

Graph  $f(x) = 5x$ ,  $f(x) = 5x^2$ , and  $f(x) = 5|x|$ . Describe how multiplying each parent function by 5 affects the corresponding parent graph.

**Solution:**

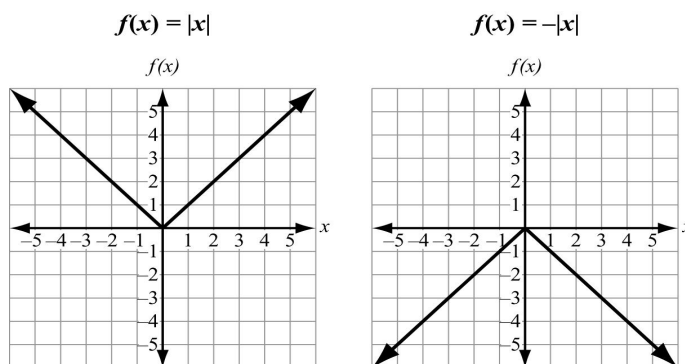
When 5 is multiplied to each parent function, the corresponding parent graphs become more narrow, otherwise known as vertically stretched.

**Example:**

Compare the graph of the parent function  $f(x) = |x|$  with the graph of  $f(x) = -|x|$ . Then determine what the graph of the parent function  $f(x) = x^2$  will look like, without graphing, when multiplied by  $-1$ :  $f(x) = -(x^2)$ .

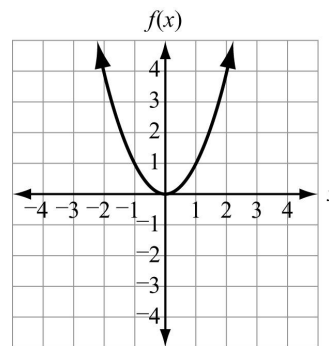
**Solution:**

Multiplying the parent function  $f(x) = |x|$  by  $-1$  reflects the parent function across the  $x$ -axis.

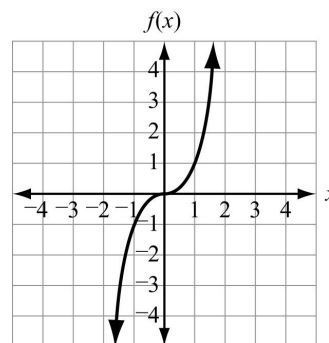


So, the graph of  $f(x) = -(x^2)$  is the reflection of the parent function  $f(x) = x^2$  reflected across the  $x$ -axis.

2. A function is an **even function** if  $f(x) = f(-x)$  for all values in its domain. The graph of an even function is **symmetric** with respect to the  $y$ -axis. This means the graph is unchanged when reflected across the  $y$ -axis. The graph to the right shows an even function.

**Even Function**

A function is an **odd function** if  $f(-x) = -f(x)$  for all values in its domain. The graph of an odd function is symmetric about the origin. This means that the graph is unchanged when reflected across both the  $x$ - and  $y$ -axes. The graph to the right shows an odd function.

**Odd Function**

3. When given a table of values, you can either examine the ordered pairs or plot the ordered pairs to see if the graph is odd, even, or neither.

**Example:**

The table at the right shows the values of a function.  
Determine whether the function is odd, even, or neither.

$x$	$f(x)$
3	9
2	4
1	1
0	0
-1	1
-2	4
-3	9

**Solution:**

Examine the ordered pairs:

$$f(3) = 9; f(-3) = 9$$

$$f(2) = 4; f(-2) = 4$$

$$f(1) = 1; f(-1) = 1$$

Since  $f(x) = f(-x)$ , the function is an even function.

**Example:**

The table at the right shows the values of a function.  
Determine whether the function is odd, even, or neither.

$x$	$f(x)$
3	27
2	8
1	1
0	0
-1	-1
-2	-8
-3	-27

**Solution:**

Examine the ordered pairs:

$$f(-3) = -27; -f(3) = -27$$

$$f(-2) = -8; -f(2) = -8$$

$$f(-1) = -1; -f(1) = -1$$

Since  $f(-x) = -f(x)$ , the function is an odd function.

4. **Function notation** provides an efficient way to define and communicate functions. A function can be described using function notation:  $f(x)$ . This notation is read out loud as “ $f$  of  $x$ .” It represents the output of the function  $f$  given the input value  $x$ .

**Example:**

The equation  $y = x^3$  written in function notation is  $f(x) = x^3$ .

5. Functions have three parts: (i) a **domain**, which is the set of inputs for the function, (ii) a **range**, which is the set of outputs, and (iii) some **rule or statement of correspondence** indicating how each input determines a unique output.

6. The domain and rule of correspondence determine the range of a function.

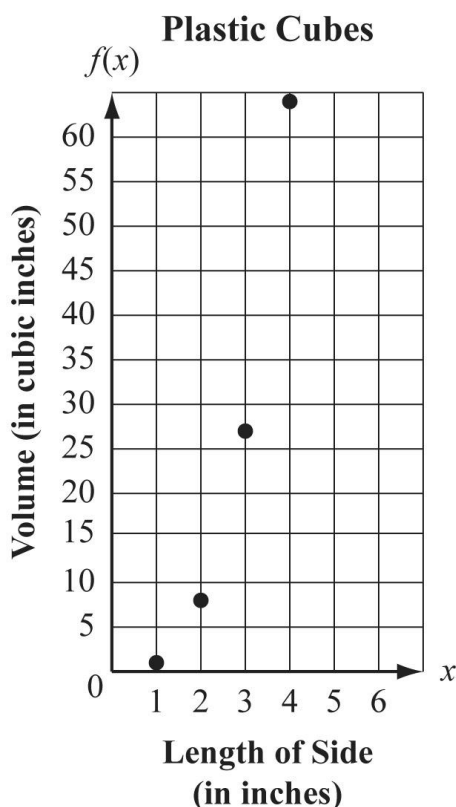
**Example:**

A company makes plastic cubes with sides that have lengths that are 1, 2, 3, or 4 inches long. The rule of correspondence for the function is  $f(x) = x^3$ , which represents the relationship between  $x$ , the length of the side of a cube, and  $f(x)$ , the volume of the cube. The domain would be {            } and the range would be {            }

7. Graphs are geometric representations of functions.

**Example:**

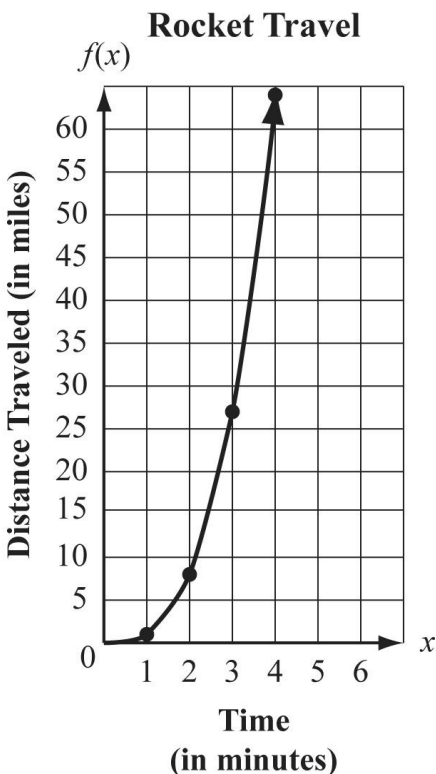
This graph represents the function  $f(x) = x^3$  from the example in Key Idea #6.



Note that the points start at 1, which is the smallest length of a side of a plastic cube. They are not connected because that would imply that the lengths of the cubes could include numbers in between the whole numbers that are given in the context of the problem. This is called a *discrete* function.

**Example:**

In one phase of a video game, a player is able to travel a distance, in miles, based on  $x$ , the number of seconds he is able to stay on a rocket. This graph represents the function  $f(x) = x^3$ , the relationship between the time, in minutes, on the rocket and the distance traveled.



Note that the graph starts at 0 and the dots are connected. Time is continuous, so the graph should be a continuous (curved) line. This is called a **continuous** function.

8. Functions are equal if they have the same domain and rule of correspondence.

**Example:**

The functions  $f(x) = x^3$  with the domain  $\{1, 2, 3, 4, 5, 6\}$  and  $g(x) = x^3$  with the domain  $\{1, 2, 3, 4, 5, 6\}$  are equal.

9. The variables used to represent domain values, range values, and the function as a whole are arbitrary. Changing variable names does not change the function.
10. Functions that are nonlinear, such as quadratic, cubic, absolute-value, or square root functions, do not have a constant rate of change. Nonlinear functions have a **variable rate of change**. This means that the change in the  $f(x)$ -value divided by the corresponding change in the  $x$ -value of a function over its entire domain is *not* always the same.



11. Consider the related functions whose relationships are specified by these equations and that each has the domain of the set of all real numbers.

(i)  $y = x^2$

(ii)  $y = 5x^2$

(iii)  $y = 6x^2$

Each of these equations has the form  $y = ax^2$ , where  $a$  is a constant real number. In each of these situations, it is said that the  $y$  *varies directly* as the square of  $x$ .

12. The *vertex* of a quadratic equation is its highest or lowest point on a graph.

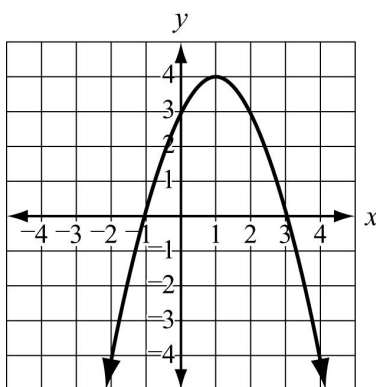


### Important Tips for Working with Functions

- Begin exploration of a new function by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.
- Do extensive graphing by hand.
- Be extremely careful in the use of language. Always use the name of the function. For example, use  $f$  to refer to the function as a whole and use  $f(x)$  to refer to the output when the input is  $x$ . For example, when language is used correctly, a graph of the function  $f$  in the  $x, y$ -plane is the graph of the equation  $y = f(x)$ , since only those points of the form  $(x, y)$  where the  $y$ -coordinates are equal to  $f(x)$  are graphed.

### REVIEW EXAMPLES

- 1) This graph represents the function  $f(x) = -x^2 + 2x + 3$ .



- Identify the domain and the range of the function.
- Identify the coordinates of the vertex. State whether the function has a maximum or minimum value.
- Identify the zeros of the function and explain what they are.

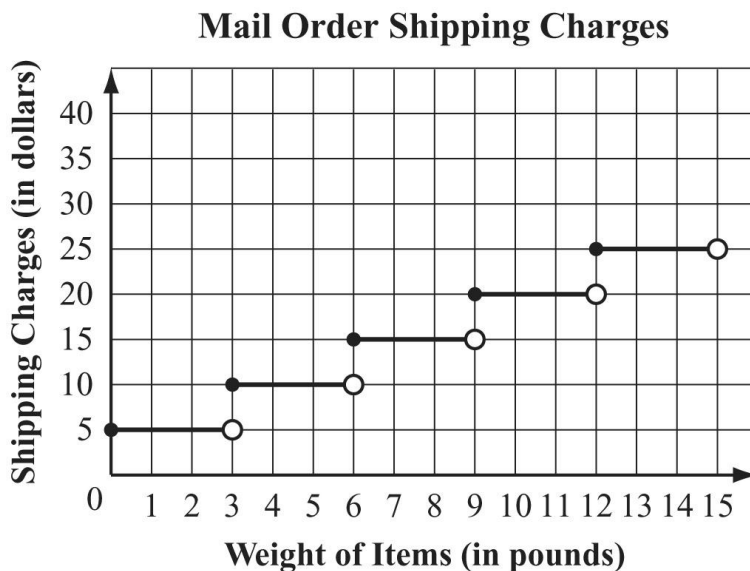
**Solution:**

- The domain is the set of all real numbers. The range is the set of numbers such that  $y \leq 4$ .
- The coordinates of the vertex are  $(1, 4)$ . The graph of the function is a parabola that opens downward, so it has a maximum value.
- The zeros are the points where the graph crosses the  $x$ -axis. It crosses at  $x = -1$  and  $x = 3$ . That means the value of the function is zero at  $x = -1$  and  $x = 3$ .

2) A mail order company charges shipping based on the total weight of all the items purchased by a customer.

- The charge to ship items that weigh less than 3 pounds is \$5.
- The charge to ship items that weigh at least 3 pounds but less than 6 pounds is \$10.
- The charge to ship items that weigh at least 6 pounds but less than 9 pounds is \$15.
- The charge to ship items that weigh at least 9 pounds but less than 12 pounds is \$20.
- The charge to ship items that weigh at least 12 pounds but less than 15 pounds is \$25.
- The pattern for charging continues.

This graph shows a function that represents the relationship between the total weight of all the items purchased by a customer and his or her shipping charges.



- What is the domain of the function? Explain what the domain is in the context of the problem.
- What is the range of the function? Explain what the range is in the context of the problem.
- What is the charge to ship items weighing a total of  $3\frac{1}{2}$  pounds?

**Solution:**

- The domain is any positive real number. The domain is the total number of pounds of the items purchased by a customer.
- The range is  $\{\$5, \$10, \$15, \$20, \$25, \dots\}$ . The range is the amount of shipping paid by a customer. Note that the  $y$ -intercept of the function has a solid dot even though you would not ship something that weighs 0 pounds.
- The charge is \$10.

3) This table shows the total number of paper airplanes Gina made after school over time.

**Gina's Paper Airplanes**

<b>Time (in minutes)</b>	0	5	10	15	20	25	30
<b>Total Number of Paper Airplanes</b>	0	5	10	15	23	31	39

- What is Gina's average rate of making paper airplanes during the first 15 minutes she made them?
- What is Gina's average rate of making paper airplanes during the last 15 minutes she made them?

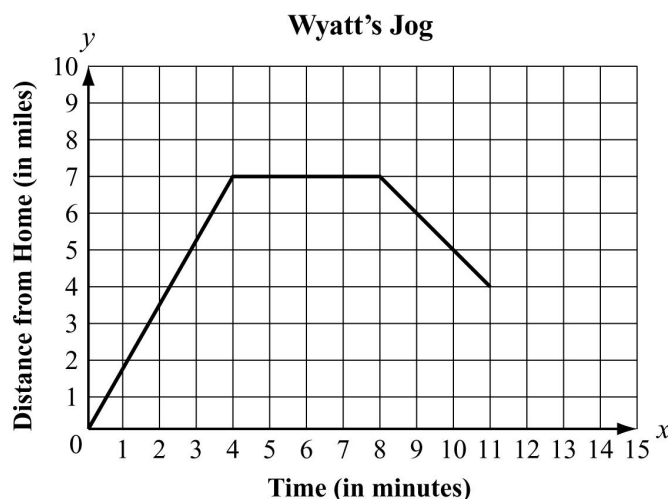
**Solution:**

- $\frac{15 - 0}{15 - 0} = \frac{15}{15} = 1$  airplane per minute.
- $\frac{39 - 15}{30 - 15} = \frac{24}{15} = 1.6$  airplanes per minute.

**Note:** This is a nonlinear function. It has a *variable rate of change*.

**EOCT Practice Items**

1) Which statement best describes what is being modeled by this graph?



- A. Wyatt started from a standstill, gradually picked up speed, jogged at a constant rate for 4 minutes, gradually slowed down, and stopped.
- B. Wyatt began jogging at a constant rate and increased his pace steadily until coming to a complete stop after jogging for 11 minutes.
- C. Wyatt jogged at a steady pace for 4 minutes, took a 4-minute break, walked at a steady pace for 3 minutes, and stopped.
- D. Wyatt jogged uphill for 4 minutes, jogged on a flat surface for 4 minutes, jogged downhill for 3 minutes, and stopped.

[Key: C]

2) Heather is taking a turn playing a game.

- If she answers the first question correctly, she is awarded 2 points.
- If she answers the second question correctly, she is awarded 4 points.
- If she answers the third question correctly, she is awarded 6 points.

Heather's turn and this pattern will continue until she is not able to answer a question correctly. Heather answers  $n$  questions correctly during her turn. Which function can be used to calculate the total number of points that she was awarded?

- A.  $f(n) = n^2 + 2n$
- B.  $f(n) = n^2 + n$
- C.  $f(n) = n^2 + 2$
- D.  $f(n) = n^2 + 1$

[Key: B]

## SEQUENCES



### KEY IDEAS

1. A **sequence** is an ordered list of numbers, pictures, letters, geometric figures, or just about any object you like. Each number, figure, or object is called a **term** in the sequence. For convenience, the terms of sequences are often separated by commas.
2. Each term in a sequence is typically represented by  $a_n$ , where  $n$  is a whole number that represents the location of that term in the sequence.

#### Example:

Consider the sequence 6, 3, 0, -3, -6, -9, .... The first term in the sequence is 6, and it is represented by  $a_1$ . The second term in the sequence is 3, and it is represented by  $a_2$ . The  $n$ th term of the sequence is represented by  $a_n$ .

3. **Finite** sequences contain a finite (bounded) number of terms.
4. **Infinite** sequences contain an infinite (unbounded) number of terms. The sequence continues in the same pattern to infinity.
5. The three dots within a list of terms in some sequences is called an **ellipsis** and indicates that some of the terms are missing. An ellipsis is necessary at the end of an infinite sequence to indicate that the sequence goes on and on to infinity.
6. Some sequences follow predictable patterns, though the pattern might not be immediately apparent. Other sequences have no pattern at all.

#### Example:

2, 4, 6, 8, 10, . . .

This sequence has a pattern. It is even numbers in increasing order.

#### Example:

7, 9, 3, 5, 4, 8

This sequence does not appear to have a pattern.

7. In looking for patterns in sequences, it is useful to look for a pattern in how each term relates to the previous one. If there is a consistent pattern in how each term relates to the previous one, it is convenient to express this pattern using a **recursive definition** for the sequence. A recursive definition gives the first term and a formula for how the  $n$ th term relates to the  $(n-1)$ th term.

**Example:**

Sequence: 3, 6, 9, 12, 15, . . .

Recursive definition:  $t_1 = 3$ ,  $t_n = t_{n-1} + 3$ 

8. Another way to define a sequence uses a **closed form definition** that indicates how to determine the  $n$ th term directly.

**Example:**

Sequence: 3, 6, 9, 12, 15, . . .

Closed form definition:  $t_n = 3n$ , for  $n = 1, 2, 3, \dots$ **REVIEW EXAMPLES**

- 1) Consider this sequence.

5, 7, 11, 19, 35, 67, . . .

- Is this a finite sequence or an infinite sequence?
- What is the domain? What is the range?
- What is the domain of the sequence? What is the range?

**Solution:**

- The ellipsis at the end of the sequence indicates that it is an infinite sequence.
- The domain is  $\{1, 2, 3, \dots\}$  is 5, the range is  $\{5, 7, 11, 19, 35, 67, \dots\}$  is 11.
- The domain is  $\{1, 2, 3, \dots\}$ , and the range is  $\{5, 7, 11, 19, 35, 67, \dots\}$

Note that this sequence has a pattern that can be expressed using the recursive definition  $a_1 = 5$ ,  $a_n = 2a_{n-1} - 3$ .

- 2) The function  $f(n) = -(1 - 4n)$  represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

**Solution:**

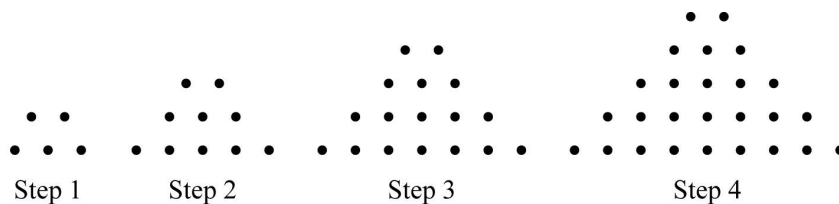
$n$	1	2	3	4	5
$f(n)$	3	7	11	15	19

Since the function is a sequence, the domain would be  $n$ , the number of each term in the sequence. The set of numbers in the domain can be written  $\{ \quad \}$ . Notice that the domain is an infinite set of numbers, even though the table lists only the first 5.

The range is  $f(n)$  or  $\quad$  the output numbers that result from applying the rule  $-(1 - 4n)$ . The set of numbers in the range, which is the sequence itself, can be written  $\{3, 7, 11, 15, 19, \dots\}$ . This is also an infinite set of numbers, even though the table lists only the first 5.

### EOCT Practice Items

- 1) These are the first four steps of a dot pattern.



The pattern continues. Which function represents the number of dots in Step  $n$ ?

- A.  $f(n) = n^2 + n - 5$   
 B.  $f(n) = n^2 + n + 3$   
 C.  $f(n) = n^2 + 5n - 1$   
 D.  $f(n) = n^2 + 2n + 2$

[Key: D]

2) The first term in this sequence is  $-3$ .

$n$	1	2	3	4	5	...
$a_n$	$-3$	4	23	60	121	...

Which function represents the sequence?

- A.  $f(n) = n^2 - 4$
- B.  $f(n) = n^3 - 4$
- C.  $f(n) = -n^3 + 4$
- D.  $f(n) = -n^2 + 4$

[Key: B]



## Unit 4: Algebra in Context

The focus of this unit is on the development of students' abilities to solve simple quadratic, rational, and radical equations using a variety of methods. Students extend and apply the skills and understandings of Units 2 and 3 through further investigation of quadratic, rational, and radical functions. The even and odd symmetry of graphs will be explored, as well as the intersections of graphs as solutions to equations.

### KEY STANDARDS

**MM1A1. Students will explore and interpret the characteristics of functions, using graphs, tables, and simple algebraic techniques.**

- c. Graph transformations of basic functions including vertical shifts, stretches, and shrinks, as well as reflections across the  $x$ - and  $y$ -axes.
- d. Investigate and explain characteristics of a function: domain, range, zeros, intercepts, intervals of increase and decrease, maximum and minimum values, and end behavior.
- h. Determine graphically and algebraically whether a function has symmetry and whether it is even, odd, or neither.
- i. Understand that any equation in  $x$  can be interpreted as the equation  $f(x) = g(x)$ , and interpret the solutions of the equation as the  $x$ -value(s) of the intersection point(s) of the graphs of  $y = f(x)$  and  $y = g(x)$ .

**MM1A3. Students will solve simple equations.**

- a. Solve quadratic equations in the form  $ax^2 + bx + c = 0$ , where  $a = 1$ , by using factorization and finding square roots where applicable.
- b. Solve equations involving radicals such as  $\sqrt{x} + b = c$ , using algebraic techniques.
- c. Use a variety of techniques, including technology, tables, and graphs to solve equations resulting from the investigation of  $x^2 + bx + c = 0$ .
- d. Solve simple rational equations that result in linear equations or quadratic equations with leading coefficient of 1.

## QUADRATIC EQUATIONS



### KEY IDEAS

1. A **quadratic equation** is an equation that can be written in the standard form  $ax^2 + bx + c = 0$ .
2. The **Addition Property of Equality** allows us to get an equivalent equation by adding the same expression to both sides of the equation. The **Multiplicative Property of Equality** allows us to get an equivalent equation by multiplying both sides of the equation by the same number as long as the number used is not 0.
3. The **Zero Product Property** states that for real numbers  $x$  and  $y$ , if  $xy = 0$ , then  $x = 0$  or  $y = 0$ , or  $x$  and  $y$  both equal 0.
4. One way to solve a quadratic equation in the form  $x^2 + bx + c = 0$  is by factoring and applying the special product  $(x + a)(x + b) = x^2 + (a + b)x + ab$ .

#### Example:

Solve the equation:  $x^2 + 5x + 6 = 0$ .

#### Solution:

$$(x + 2)(x + 3) = 0$$

Factor.

$$x + 2 = 0 \qquad x + 3 = 0$$

Use the Zero Product Property to set each factor equal to 0.

$$x + 2 - 2 = 0 - 2 \qquad x + 3 - 3 = 0 - 3$$

Use the properties of equality to solve each equation.

$$x = -2 \qquad \text{or} \qquad x = -3$$

There are two solutions.

The answer can be checked by substituting it into the original equation and simplifying. The statement will be true if the answer is correct.

#### Check $x = -2$

$$(-2)^2 + 5(-2) + 6 = 0$$

$$4 - 10 + 6 = 0$$

$$0 = 0$$

This is true, so  $-2$  is a solution.

#### Check $x = -3$

$$(-3)^2 + 5(-3) + 6 = 0$$

$$9 - 15 + 6 = 0$$

$$0 = 0$$

This is true, so  $-3$  is a solution.

5. One way to solve a quadratic equation in the form  $x^2 - c = 0$ , where the value of  $c$  is a perfect square, is to use the *difference of squares* identity.

**Example:**

$$\begin{aligned} x^2 - 36 &= 0 \\ (x-6)(x+6) &= 0 \\ x-6 &= 0 & x+6 &= 0 \\ x-6+6 &= 0+6 & x+6-6 &= 0-6 \\ x &= 6 & \text{or} & x &= -6 \end{aligned}$$

6. A quadratic equation in the form  $x^2 - c = 0$  always has two solutions for  $c > 0$ ,  $\sqrt{c}$  and  $-\sqrt{c}$ . This is true whether the value of  $c$  is a perfect square or not.

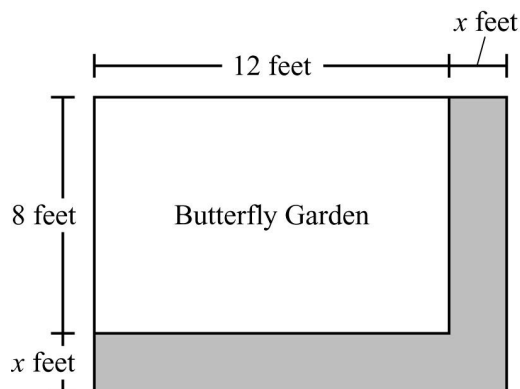
**Example:**

$$\begin{aligned} x^2 - 7 &= 0 \\ x &= \sqrt{7} \text{ or } x = -\sqrt{7} \end{aligned}$$

7. There is an important distinction between solving an equation and solving an applied problem modeled by an equation. The situation that gave rise to the equation may include restrictions on the solution to the applied problem that eliminate certain solutions to the equation.

**REVIEW EXAMPLES**

- 1) Carrie has a rectangular butterfly garden that is 12 feet long by 8 feet wide. She wants to put a sidewalk along two sides of the garden, as shown by the shaded area of this diagram.



Carrie has enough concrete for the sidewalk to cover 44 square feet. What is the maximum width she can make her sidewalk?

**Solution:**

One way to do this is to divide the shaded area into two rectangles. Represent the area of each rectangle algebraically, add them together, and set them equal to 44, the number of square feet Carrie can cover with concrete.

$$12x + x(8 + x) = 44$$

$$12x + 8x + x^2 = 44 \quad \text{Simplify.}$$

$$x^2 + 20x = 44 \quad \text{Simplify.}$$

$$x^2 + 20x - 44 = 0 \quad \text{Use algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in terms of } x \text{ on one side of the equation and } 0 \text{ on the other.}$$

$$(x - 2)(x + 22) = 0 \quad \text{Factor.}$$

$$x - 2 = 0 \quad x + 22 = 0 \quad \text{Set each factor equal to } 0.$$

$$x - 2 + 2 = 0 + 2 \quad x + 22 - 22 = 0 - 22 \quad \text{Solve each equation.}$$

$$x = 2 \quad \text{or} \quad x = -22$$

Since the question asks for the width of a sidewalk, the first solution of 2 feet makes sense. The second solution of  $-22$  feet does not make sense, so eliminate it. Therefore, the maximum width Carrie can make her sidewalk is 2 feet.

- 2) A right triangle has one leg that is 9 centimeters long. Its hypotenuse is 10 centimeters long. What is the length, in centimeters, of the other leg?

**Solution:**

Represent the unknown side with a variable, such as  $a$ . Then use the Pythagorean theorem to set up a relationship.

$$a^2 + 9^2 = 10^2$$

$$a^2 + 81 = 100 \quad \text{Simplify.}$$

$$a^2 + 81 - 100 = 100 - 100 \quad \text{Subtract } 100 \text{ from each side.}$$

$$a^2 - 19 = 0 \quad \text{Simplify.}$$

$$a = \sqrt{19} \quad \text{or} \quad a = -\sqrt{19}$$

The length of the other leg of the triangle is  $\sqrt{19}$  centimeters. Since this solution represents a length, take the square root to get approximately 4.4 centimeters for the answer.

Note that  $-\sqrt{19}$  is eliminated as an answer because a length cannot be negative.

**EOCT Practice Items**

- 1) A squirrel in a tree dropped an acorn 48 feet to the ground. The number of seconds,  $t$ , it took the acorn to reach the ground is modeled by this equation.

$$-16t^2 + 48 = 0$$

How many seconds did it take the acorn to reach the ground?

- A.  $\sqrt{\quad}$
- B. 3
- C.  $\sqrt{\quad}$
- D. 32

[Key: A]

- 2) An art teacher painted a rectangular picture on the art room wall. Then she enlarged it by increasing both the width and the length by  $x$  feet. This equation can be solved to find  $x$ , the number of feet the art teacher increased each dimension of her picture.

$$x^2 + 7x - 18 = 0$$

The area of the enlarged picture was 35 square feet. Which dimensions could be for the picture before it was enlarged?

- A. 3 feet by 6 feet
- B. 5 feet by 3 feet
- C. 7 feet by 5 feet
- D. 9 feet by 2 feet

[Key: B]

## RATIONAL EQUATIONS



### KEY IDEAS

1. An equation that contains one or more rational expressions is called a *rational equation*.
2. Techniques for solving rational equations include steps that may give extraneous solutions that do not solve the original rational equation. Check each answer in the original problem to find and eliminate any extraneous solutions.
3. To solve a rational equation that is composed of one rational expression equal to another rational expression, find the common denominator and multiply both sides of the equation by it.

#### Example:

$$\frac{4}{x+3} = \frac{7}{x}$$

The common denominator is  $x(x+3)$ .

$$(x)(x+3)\left(\frac{4}{x+3}\right) = (x)(x+3)\left(\frac{7}{x}\right)$$

Multiply both sides by the common denominator.

$$4(x) = 7(x+3)$$

Simplify.

$$4x = 7x + 21$$

Simplify.

$$4x - 7x = 7x + 21 - 7x$$

Subtract  $7x$  from each side.

$$-3x = 21$$

Simplify.

$$\frac{-3x}{-3} = \frac{21}{-3}$$

Divide each side by  $-3$ .

$$x = -7$$

Check the solution by substituting it into the original problem.

$$\frac{4}{x+3} = \frac{7}{x}$$

Original problem.

$$\frac{4}{-7+3} = \frac{7}{-7}$$

Substitute answer into problem.

$$\frac{4}{-4} = \frac{7}{-7}$$

Simplify.

$$-1 = -1$$

Simplify.

4. To solve a rational equation that is composed of two or more rational expressions being combined on the same side of the equation, find the common denominator and build the expressions so the terms have common denominators. Then, combine the rational expressions.

**Example:**

What is the solution to the equation  $\frac{3}{a} + \frac{1}{2a} = 9$ ?

**Solution:**

One way to do this is to find the common denominator and add the fractions to the left of the equal sign.

$$\frac{3}{a} + \frac{1}{2a} = 9$$

The common denominator for the fractions is  $2a$ .

$$\frac{3}{a} \left( \frac{2}{2} \right) + \frac{1}{2a} = 9$$

Multiply the first fraction by a fraction that equals one to build a common denominator. (The second fraction already has the common denominator.)

$$\frac{6}{2a} + \frac{1}{2a} = 9$$

Multiply.

$$\frac{7}{2a} = 9$$

Combine the numerators but keep the denominator the same.

$$(2a) \left( \frac{7}{2a} \right) = (2a)(9)$$

Multiply both sides of the equation by  $2a$ .

$$7 = 18a$$

$$\frac{7}{18} = \frac{18a}{18}$$

Divide both sides of the equation by 18.

$$a = \frac{7}{18}$$

Check the solution by substituting it into the original problem.

$$\frac{3}{\frac{7}{18}} + \frac{1}{2 \left( \frac{7}{18} \right)} = 9$$

$$\frac{54}{7} + \frac{18}{14} = 9$$

$$\frac{54}{7} + \frac{9}{7} = 9$$

$$\frac{63}{7} = 9$$

$$9 = 9$$

**REVIEW EXAMPLES**

1) Jenna had 10 red marbles and 20 blue marbles in a bag.

- She added  $x$  red marbles and  $x$  blue marbles to the bag.
- She took one-fourth of the red marbles and one-half of the blue marbles out of the bag.
- The total number of marbles she took out of the bag was 17.

This equation can be used to find  $x$ , the number of red marbles or the number of blue marbles that Jenna took out of the bag.

$$\frac{x+10}{4} + \frac{x+20}{2} = 17$$

What was the total number of red and blue marbles that were left in the bag?

**Solution:**

$\frac{x+10}{4} + \frac{x+20}{2} = 17$	The LCD is 4.
$4\left(\frac{x+10}{4}\right) + 4\left(\frac{x+20}{2}\right) = 4(17)$	Multiply through by the LCD.
$x+10+2x+40=68$	Simplify.
$3x+50=68$	Simplify.
$3x+50-50=68-50$	Subtract 50 from both sides of the equation.
$3x=18$	Simplify.
$x=6$	

The value of  $x$  is 6. This means she had  $10 + 6$ , or 16, blue marbles, and  $20 + 6$ , or 26, red marbles before she removed any. That was a total of  $16 + 26$ , or 42 marbles. Then she removed 17 marbles, so she had  $42 - 17$ , or 25 red and blue marbles left in the bag.



2) What value of  $x$  makes this equation true?

$$\frac{4}{x-2} = \frac{3}{2x+1}$$

**Solution:**

$$\frac{4}{x-2} = \frac{3}{2x+1}$$

$$(x-2)(2x+1)\left(\frac{4}{x-2}\right) = (x-2)(2x+1)\left(\frac{3}{2x+1}\right) \quad \text{Multiply both sides by the LCD.}$$

$$4(2x+1) = 3(x-2) \quad \text{Simplify.}$$

$$8x+4 = 3x-6 \quad \text{Simplify.}$$

$$8x+4-4 = 3x-6-4 \quad \text{Subtract 4 from both sides.}$$

$$8x-3x = 3x-10-3x \quad \text{Subtract 3x from both sides.}$$

$$5x = -10 \quad \text{Simplify.}$$

$$x = -2$$

The solution is  $x = -2$ . Check to be sure that  $-2$  does not result in 0 in the denominator. If the solution results in 0 when substituted into the original equation,  $-2$  is not a valid solution.

**EOCT Practice Items**

- 1) Breanne is rowing a boat at a rate of 5 miles per hour. She can row 7 miles downstream, with the current, in the same amount of time it takes her to row 3 miles upstream, against the current. This equation can be used to find the speed of the current in the stream.

$$\frac{7}{5+c} = \frac{3}{5-c}$$

What is the speed of the current in the stream?

- A. 2 miles per hour
- B. 3 miles per hour
- C. 4 miles per hour
- D. 5 miles per hour

[Key: A]

- 2) What value of  $x$  makes this equation true?

$$\frac{4}{x} + \frac{1}{2x} = 8$$

- A. —
- B. —
- C. —
- D. —

[Key: B]

## RADICAL EQUATIONS



### KEY IDEAS

1. An equation that has a radical with a variable under the radicand is called a *radical equation*.
2. If the equation  $x = y$  is true, then the equation  $x^2 = y^2$  is true.
3. To solve a radical equation in the form  $\sqrt{x} + b = c$ , isolate the radical expression on one side of the equation. Then square both sides to eliminate the radical symbol.

#### Example:

Solve the equation  $\sqrt{x} - 15 = 3$ .

#### Solution:

$$\sqrt{x} - 15 + 15 = 3 + 15 \quad \text{Add 15 to both sides of the equation.}$$

$$\sqrt{x} = 18$$

$$(\sqrt{x})^2 = (18)^2 \quad \text{Square both sides of the equation.}$$

$$x = 324$$

The solution is  $x = 324$ .

Check your answer by substituting it back into the original equation.

$$\sqrt{x} - 15 = 3$$

$$\sqrt{324} - 15 = 3$$

$$18 - 15 = 3$$

$$3 = 3 \quad \text{This is a true equation, so the answer must be correct.}$$

4. When both sides of an equation are squared in the process of solving it, the result may be extraneous solutions that do not solve the original equation. Check each answer in the original problem to eliminate any extraneous solutions.

**REVIEW EXAMPLES**

1) Solve the equation  $15 = 6 + \sqrt{x}$ .

**Solution:**

$$15 = 6 + \sqrt{x}$$

$$15 - 6 = 6 + \sqrt{x} - 6 \quad \text{Isolate the radical.}$$

$$9 = \sqrt{x}$$

$$(9)^2 = (\sqrt{x})^2 \quad \text{Square both sides.}$$

$$x = 81$$

**Check:**

$$15 = 6 + \sqrt{x}$$

$$15 = 6 + \sqrt{81}$$

$$15 = 6 + 9$$

$$15 = 15$$

Since 81 satisfies the original equation, then 81 is the solution.

2) Solve the equation  $\sqrt{4x} - 9 = 25$ .

**Solution:**

$$\sqrt{4x} - 9 = 25$$

$$\sqrt{4x} - 9 + 9 = 25 + 9 \quad \text{Isolate the radical.}$$

$$\sqrt{4x} = 34$$

$$(\sqrt{4x})^2 = (34)^2 \quad \text{Square both sides.}$$

$$4x = 1156$$

$$\frac{4x}{4} = \frac{1156}{4} \quad \text{Divide to isolate } x.$$

$$x = 289$$

**Check:**

$$\sqrt{4x} - 9 = 25$$

$$\sqrt{4(289)} - 9 = 25$$

$$\sqrt{1156} - 9 = 25$$

$$34 - 9 = 25$$

$$25 = 25$$

Since 289 satisfies the original equation, then 289 is the solution.

3) Solve the equation  $\sqrt{b} + 5 = 2$ .

**Solution:**

$$\sqrt{b} + 5 = 2$$

$$\sqrt{b} + 5 - 5 = 2 - 5 \quad \text{Isolate the radical.}$$

$$\sqrt{b} = -3$$

$$(\sqrt{b})^2 = (-3)^2 \quad \text{Square both sides.}$$

$$b = 9$$

**Check:**

$$\sqrt{b} + 5 = 2$$

$$\sqrt{9} + 5 \neq 2$$

$$3 + 5 \neq 2$$

$$8 \neq 2$$

Since 9 does not satisfy the original equation, then 9 is not a solution. This equation has no real solution.

**EOCT Practice Items**

- 1) The base of a triangle is represented by  $\sqrt{x}$  inches. The height is 4 inches. The area of the triangle is 18 square inches.

What is the value of  $x$ ?

- A. 9
- B. 18
- C. 36
- D. 81

[Key: D]

- 2) The equation  $\sqrt{3x+4} = 5$  compares the areas of two congruent triangles. What is the value of  $x$ ?

- A. 1
- B. 3
- C. 7
- D. 25

[Key: C]

## CHARACTERISTICS OF FUNCTIONS AND THEIR GRAPHS



### KEY IDEAS

1. We call  $f$  an **even function** if  $f(x) = f(-x)$  for all values in its domain.

**Example:**

Suppose  $f$  is an even function and the point  $(2, 7)$  is on the graph of  $f$ . Name one other point that must be on the graph of  $f$ .

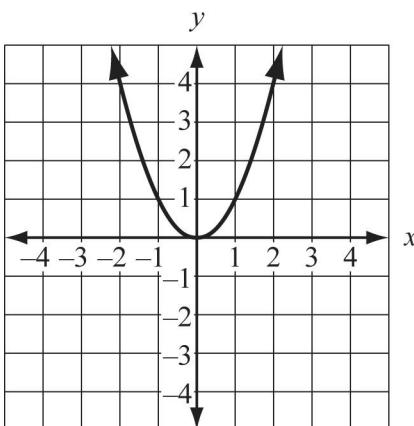
Since  $(2, 7)$  is on the graph, 2 is in the domain and  $f(2) = 7$ . By definition of an even function,  $f(-2) = f(2) = 7$ .

Therefore,  $(-2, 7)$  is also on the graph of  $f$ .

2. The graph of an even function has line symmetry with respect to the  $y$ -axis.

**Example:**

This is the graph of the function  $y = x^2$ , which is an even function.



Notice that for any number  $b$ , the points  $(x, b)$  and  $(-x, b)$  are at the same height on the grid and are equidistant from the  $y$ -axis. That means they represent line symmetry with respect to the  $y$ -axis.

3. We call  $f$  an **odd function** if  $f(-x) = -f(x)$  for all values in its domain.

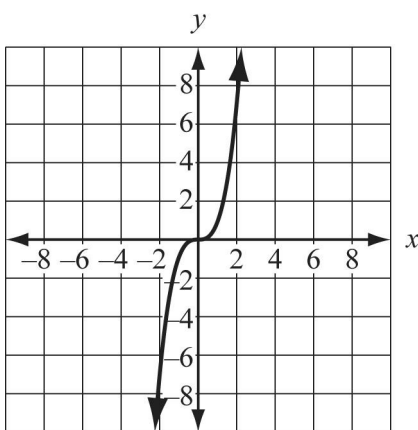
**Example:**

Suppose  $f$  is an odd function and the point  $(-2, 8)$  is on the graph of  $f$ . Name one other point that must be on the graph of  $f$ .

Since  $(-2, 8)$  is on the graph,  $-2$  is in the domain and  $f(-2) = 8$ . By definition of an odd function,  $-(-2)$ , or  $2$ , is also in the domain and  $f(2) = -f(-2) = -8$ .

Therefore,  $(2, -8)$  is also on the graph of  $f$ .

4. This is the graph of the function  $y = x^3$ , which is an odd function.



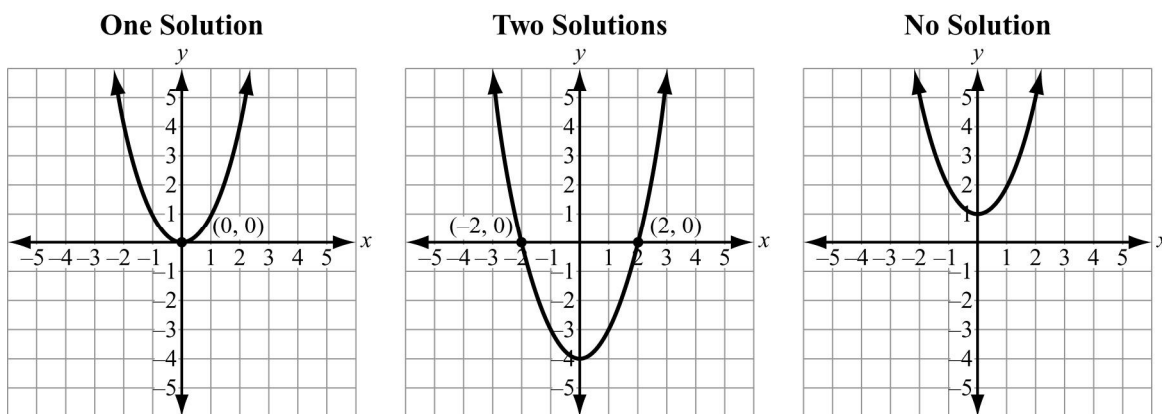
The graph of an odd function has rotational symmetry of  $180^\circ$  about the origin. This is also called symmetry with respect to the origin. Whenever the graph of an odd function contains the point  $(a, b)$  it also contains the point  $(-a, -b)$ .

5. For any graph, **rotational symmetry** of  $180^\circ$  about the origin is the same as point symmetry of reflection through the origin. Reflecting a point through the  $x$ -axis and then through the  $y$ -axis gives the same final point as rotating it  $180^\circ$  about the origin.
6. One way to solve a quadratic equation is to graph the corresponding quadratic function on a coordinate plane. The standard form of a quadratic equation is  $ax^2 + bx + c = 0$ . The corresponding function is  $f(x) = ax^2 + bx + c$ . The **solutions**, or **roots**, of the equation  $ax^2 + bx + c = 0$  can be determined by finding the values of  $x$  where the graph of the function crosses the  $x$ -axis, otherwise known as the  $x$ -intercepts. Note that in GPS Algebra, the only quadratics being addressed are those where  $a = 1$ .



A quadratic equation may have one solution, two solutions, or no solution.

- If the graph crosses the  $x$ -axis at only one point (has only one  $x$ -intercept), then the equation has only one unique solution. This is also known as a double root because the factors of the quadratic are identical and produce the same value.
- If the graph crosses the  $x$ -axis at two points (has two  $x$ -intercepts), then the equation has two solutions.
- If the graph does **not** cross the  $x$ -axis at any point (has no  $x$ -intercept), then the equation has no solution.

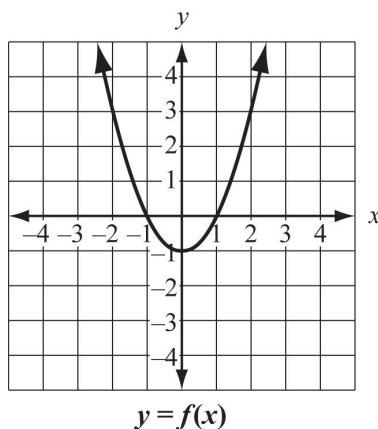


7. The graph of an equation can be reflected through the  $x$ -axis or the  $y$ -axis. If an equation is in the form  $y = f(x)$ , then:

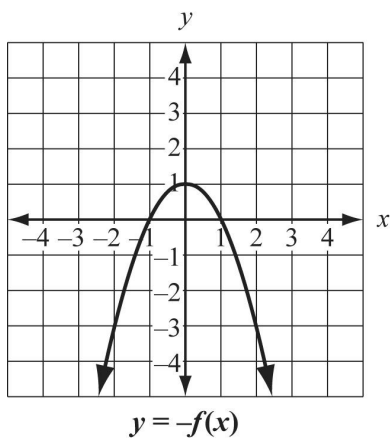
- to reflect the graph of  $y = f(x)$  through the  $x$ -axis, form the equation  $y = -f(x)$ .
- to reflect the graph of  $y = f(x)$  through the  $y$ -axis, form the equation  $y = f(-x)$ .

**Example:**

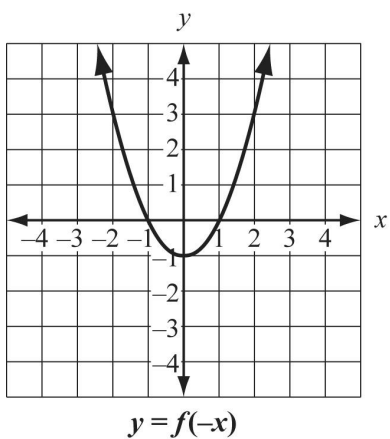
Consider the equation  $y = x^2 - 1$ . Let  $f(x) = x^2 - 1$  and then  $y = f(x)$ . This is the graph of  $y = f(x)$ .



This is the graph of the reflection of the equation  $y = f(x)$  through the  $x$ -axis.



This is the graph of the reflection of the equation  $y = f(x)$  through the  $y$ -axis.



Notice that this reflection looks the same as the original graph because the original graph was symmetric with respect to the  $y$ -axis.

**REVIEW EXAMPLES**

1) Solve the equation  $x^2 - 3x - 10 = 0$  by graphing.

**Solution:**

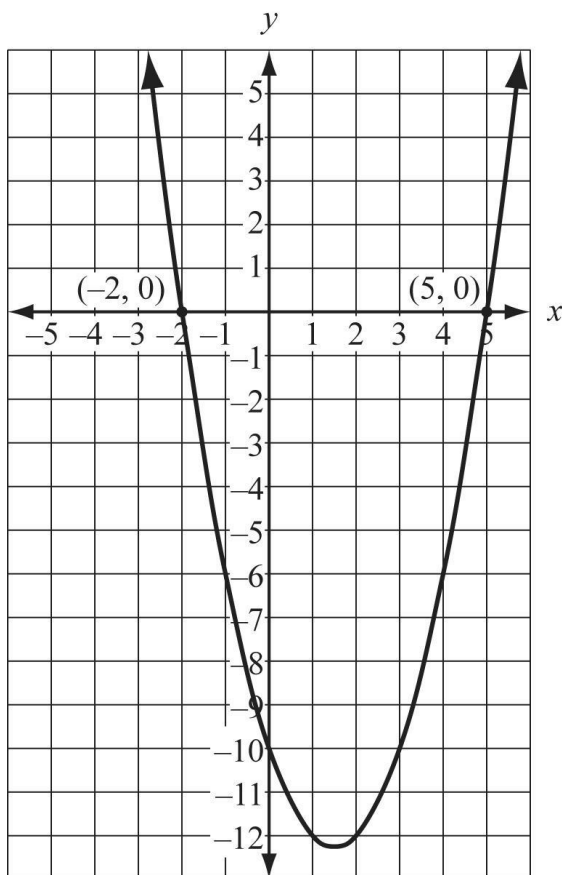
Rewrite the equation in function form.

$$y = x^2 - 3x - 10$$

Set up an  $x/y$  table for the function.

$x$	-3	-2	-1	0	1	2	3	4	5
$y$	8	0	-6	-10	-12	-12	-10	-6	0

Graph the function from the table.



The points where the graph crosses the  $x$ -axis are  $(-2, 0)$  and  $(5, 0)$ . The numbers  $-2$  and  $5$  are solutions to the quadratic equation  $x^2 - 3x - 10 = 0$  because when each is substituted for  $x$ , the result is 0.

2) Solve the equation  $x^2 + 7x + 12 = 0$ .

**Solution:**

Rewrite the equation in function form.

$$y = x^2 + 7x + 12$$

Set up an  $x/y$  table for the function.

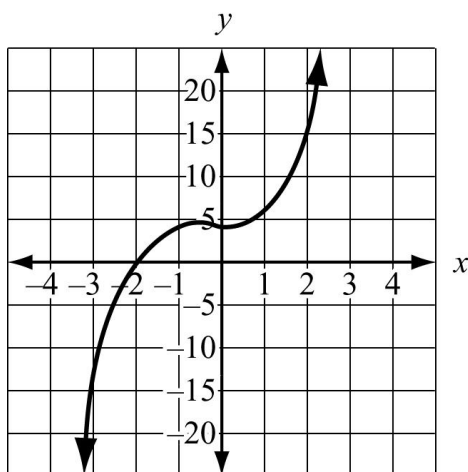
<b>x</b>	-5	-4	-3	-2	-1	0
<b>y</b>	2	0	0	2	6	12

A quadratic equation can have zero, one, or two solutions. A solution is a point where the graph of the equation crosses the  $x$ -axis. In the table, that is a point that has a  $y$ -value of 0. In this example, the points where the graph crosses the  $x$ -axis are  $(-4, 0)$  and  $(-3, 0)$ . The numbers  $-4$  and  $-3$  are solutions to the quadratic equation  $x^2 + 7x + 12 = 0$ .

Be sure to select enough values for  $x$  so that you can see the pattern in the table where the parabola formed by the quadratic equation changes direction. Select enough points on either side of the turning point to see whether the graph crosses the  $x$ -axis zero, one, or two times.

**EOCT Practice Items**

- 1) The graph of the function  $f(x) = x^3 + x^2 + 4$  is shown on this coordinate plane.



Which statement best describes the behavior of the function within the interval  $x = -3$  to  $x = 0$ ?

- A. From left to right, the function rises only.
- B. From left to right, the function falls and then rises.
- C. From left to right, the function rises and then falls.
- D. From left to right, the function falls, rises, and then falls.

[Key: C]

- 2) What is the formula for the function with a graph the same as the graph obtained by reflecting the graph of  $y = \sqrt{-x}$  across the  $x$ -axis?

- A.  $y = -\sqrt{-x}$
- B.  $y = -\sqrt{x}$
- C.  $y = \sqrt{-x}$
- D.  $y = \sqrt{x}$

[Key: A]

## Unit 5: Quadratics and Complex Numbers

This unit extends the study of quadratic functions to include in-depth analysis of general quadratic functions in both the standard form  $f(x) = ax^2 + bx + c$  and in the vertex form  $f(x) = a(x - h)^2 + k$ . Strategies for finding solutions to quadratic equations are extended to include solving them through factoring and using the quadratic formula, which can be used to solve any quadratic equation. Study of the quadratic formula introduces complex numbers; the arithmetic of complex numbers also is introduced and explored. Connections are made between algebraic results and characteristics of the graphics of quadratic functions. These connections are used to solve quadratic equations and inequalities. In addition, sums of terms of finite arithmetic sequences are explored as examples of a quadratic function. Interval notation is commonly used for representing an interval (such as a domain, range, or solution set) as a pair of numbers. This work provides a foundation for modeling data with quadratic functions, a topic that will be explored in Unit 6.

### KEY STANDARDS

**MM2N1. Students will represent and operate with complex numbers.**

- Write square roots of negative numbers in imaginary form.
- Write complex numbers in the form  $a + bi$ .
- Add, subtract, multiply, and divide complex numbers.
- Simplify expressions involving complex numbers.

**MM2A3. Students will analyze quadratic functions in the forms  $f(x) = ax^2 + bx + c$  and  $f(x) = a(x - h)^2 + k$ .**

- Convert between standard and vertex form.
- Graph quadratic functions as transformations of the function  $f(x) = x^2$ .
- Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.
- Explore arithmetic series and various ways of computing their sums.
- Explore sequences of partial sums of arithmetic series as examples of quadratic functions.

**MM2A4. Students will solve quadratic equations and inequalities in one variable.**

- Solve equations graphically using appropriate technology.
- Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
- Analyze the nature of roots using technology and using the discriminant.
- Solve quadratic inequalities both graphically and algebraically, and describe the solutions using linear inequalities.

## QUADRATIC FUNCTIONS



### KEY IDEAS

1. Quadratic functions can be located on a coordinate plane by horizontal and vertical shifts of the graph of the function  $f(x) = x^2$ .

A **horizontal shift** of  $h$  units is represented by the function  $f(x) = (x - h)^2$ . Note that if  $h$  is positive, i.e., the graph is shifted to the right,  $h$  is subtracted from  $x$ . If  $h$  is negative, i.e., the graph is shifted to the left,  $h$  is added to  $x$ .

A **vertical shift** of  $k$  units is represented by the function  $f(x) = x^2 + k$ . In this case, the direction of the shift agrees with the operation in the function.

Horizontal and vertical shifts can be combined. If the graph of  $f(x) = x^2$  is translated so its vertex is at the point  $(h, k)$ , it is represented by the function  $f(x) = (x - h)^2 + k$ .

2. The factor of  $x^2$  represents the amount of **vertical stretch** or **shrink** applied to the graph of  $f(x) = x^2$ . This factor also determines whether the graph opens up, i.e., has a vertex with a  $y$ -coordinate that represents a **minimum** value of  $f(x)$ , or opens down, i.e., has a vertex with a  $y$ -coordinate that represents a **maximum** value of  $f(x)$ .

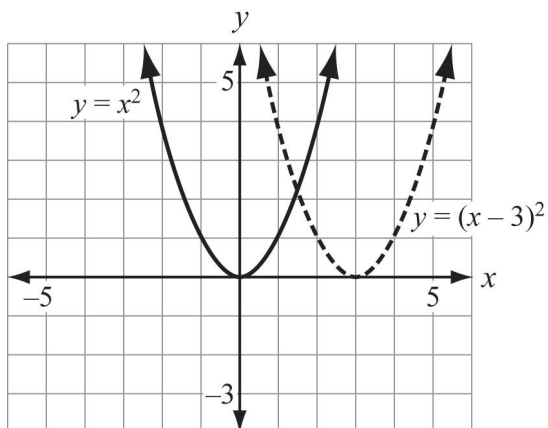
The general equation for a function with a vertex at the point  $(h, k)$  and a vertical stretch factor of  $a$  is represented by the function  $f(x) = a(x - h)^2 + k$ .

#### Example:

- a. Graph  $y = x^2$  and  $y = (x - 3)^2$ .
- b. Make a table to show the values of  $x$  and  $y$  for both equations for  $x = -2, -1, 0, 1, 2, 3, 4,$  and  $5$ .
- c. Describe how subtracting 3 from the value of  $x$  in the parent function  $y = x^2$  affects the graph of  $y = (x - 3)^2$ .

**Solution:**

a.



b.

$x$	$x^2$	$(x-3)^2$
-2	4	25
-1	1	16
0	0	9
1	1	4
2	4	1
3	9	0
4	16	1
5	25	4

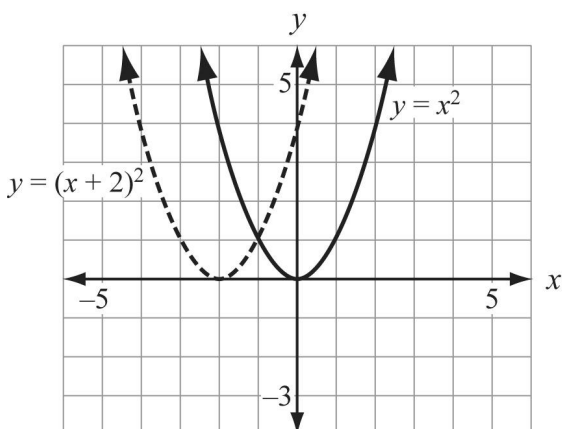
- c. When 3 is subtracted from the value of  $x$ , the values of  $(x-3)^2$  are displaced by 3 units from the values of  $x^2$ . Therefore, the graph is shifted 3 units to the right.

**Example:**

- Graph  $y = x^2$  and  $y = (x+2)^2$ .
- Make a table to show the values of  $x$  and  $y$  for both equations for  $x = -4, -3, -2, -1, 0, 1, 2,$  and  $3$ .
- Describe how adding 2 to the value of  $x$  in the parent function  $y = x^2$  affects the graph of  $y = (x+2)^2$ .

**Solution:**

a.



b.

$x$	$x^2$	$(x+2)^2$
-4	16	4
-3	9	1
-2	4	0
-1	1	1
0	0	4
1	1	9
2	4	16
3	9	25

- c. When 2 is added to the value of  $x$ , the values of  $(x+2)^2$  are displaced by 2 units from the values of  $x^2$ . Therefore, the graph is shifted 2 units to the left.

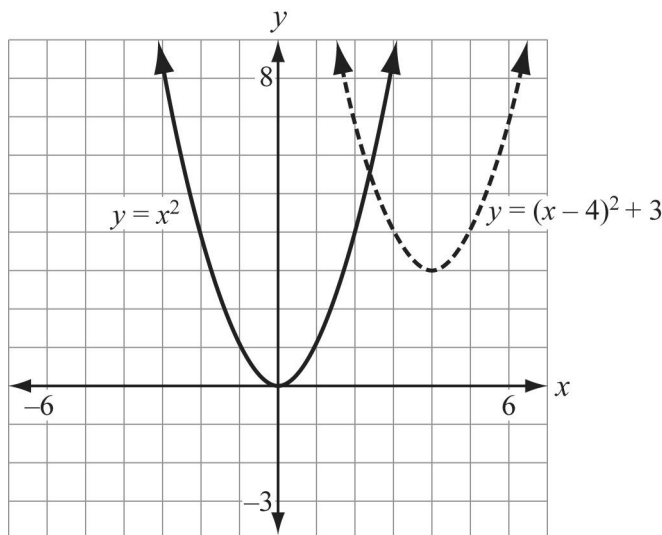


**Example:**

- Graph  $y = x^2$  and  $y = (x - 4)^2 + 3$ .
- Describe how the graph of  $y = x^2$  is translated to get the graph of  $y = (x - 4)^2 + 3$ .

**Solution:**

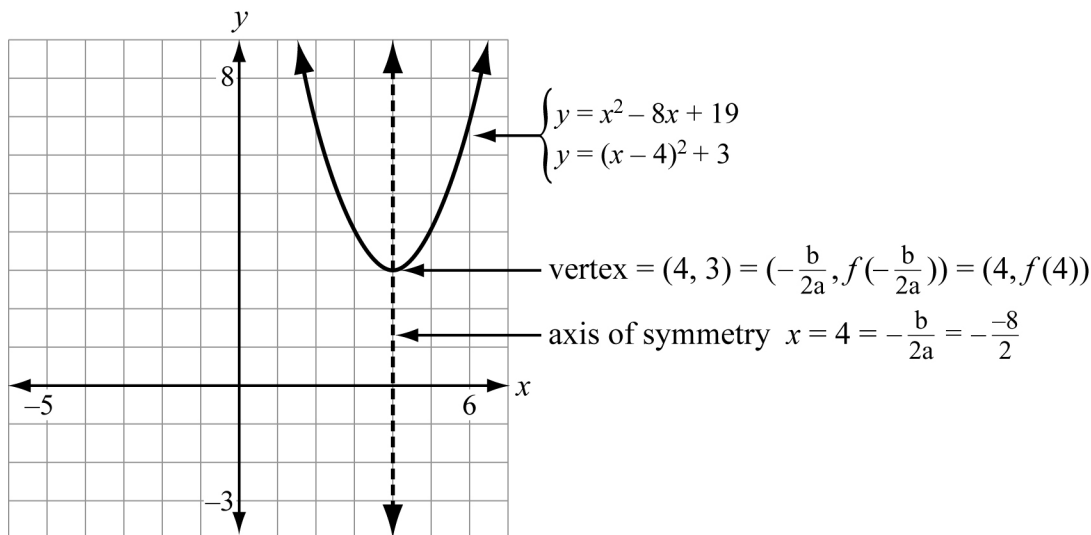
a.



- The graph of  $y = x^2$  is translated by 4 units to the right and 3 units up.
- The **vertex** and **axis of symmetry** of a quadratic function can be determined directly from the representation of the function in the form  $f(x) = a(x - h)^2 + k$ . The vertex is located at the point  $(h, k)$  and the axis of symmetry is the line  $x = h$ .
  - For a function represented in standard form as  $f(x) = ax^2 + bx + c$ , the vertex is located at the point  $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$  and the axis of symmetry is the line  $x = \frac{-b}{2a}$ .

**Example:**

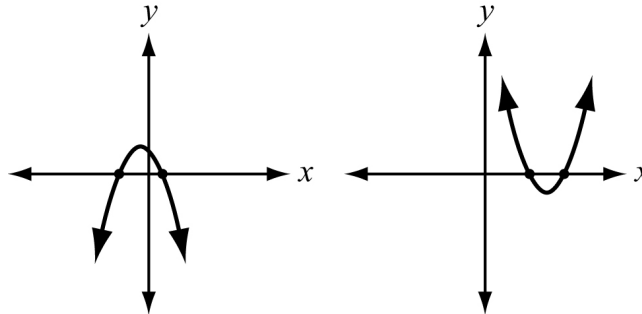
Sketch the graph of the function  $y = x^2 - 8x + 19$  on the coordinate plane. Identify the vertex and the axis of symmetry of the function. Write the same function in the form  $y = (x - h)^2 + k$ .

**Solution:**

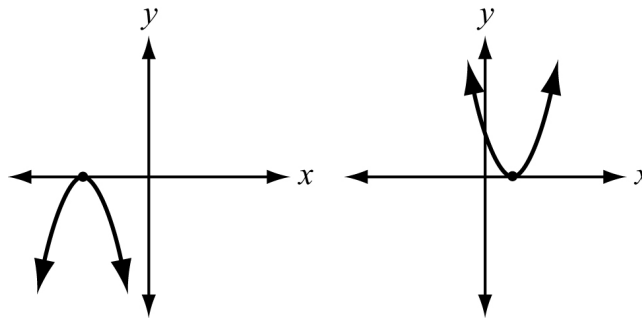
5. Every quadratic function has either a **maximum** or a **minimum value** at the vertex. For a quadratic function where  $a$  is positive, the value of  $f(x)$  at the vertex represents the minimum value of the function. From the left of the vertex to the vertex, the function *decreases at a decreasing rate*. From the right of the vertex to the vertex, the function *increases at an increasing rate*. The rate of increase or decrease is determined by the absolute value of  $a$ . For a quadratic function where  $a$  is negative, the value of  $f(x)$  at the vertex represents the maximum value of the function. To the left of the vertex, the function *increases at a decreasing rate*. To the right of the vertex, the function *decreases at an increasing rate*. The rate of increase or decrease is determined by the absolute value of  $a$ .
6. The  $x$ -intercept or intercepts of a quadratic function are also called the **zeros** of the function. This is because the value of  $f(x)$  is 0 at those points. A quadratic equation may have 0, 1, or 2 real zeros. The zeros on the graph correspond to the **real solutions** of the quadratic equation when it is set equal to 0.

**Example:**

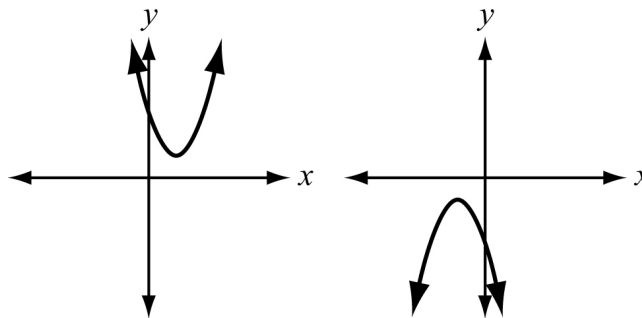
**2 Real Solutions**



**Exactly 1 Real Solution**



**No Real Solutions**

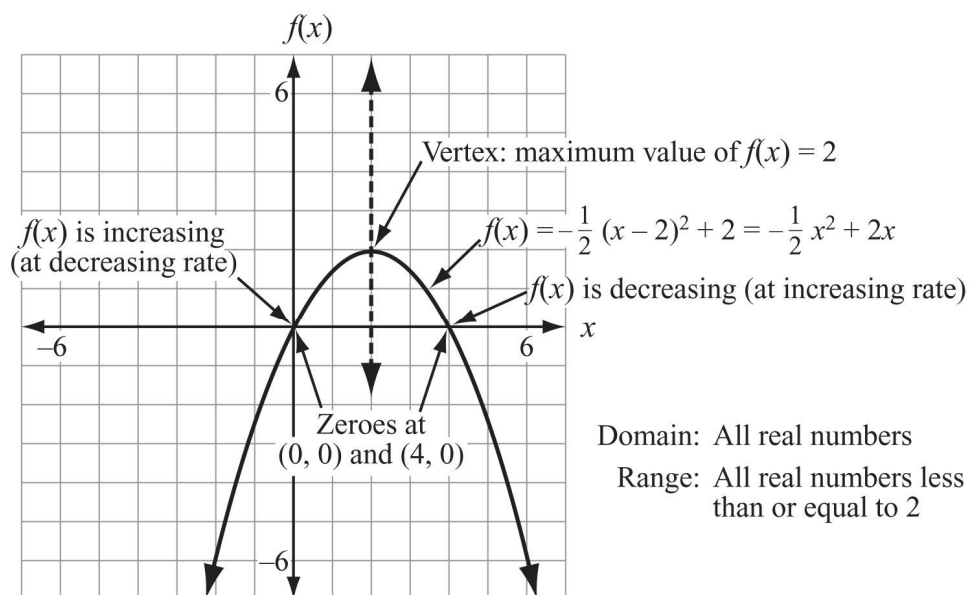


**Example:**

Sketch the graph of  $f(x) = -\frac{1}{2}(x-2)^2 + 2$  on the coordinate plane. Identify the vertex, the maximum or minimum value, the increasing and decreasing behavior, and the zeros of  $f(x)$ .

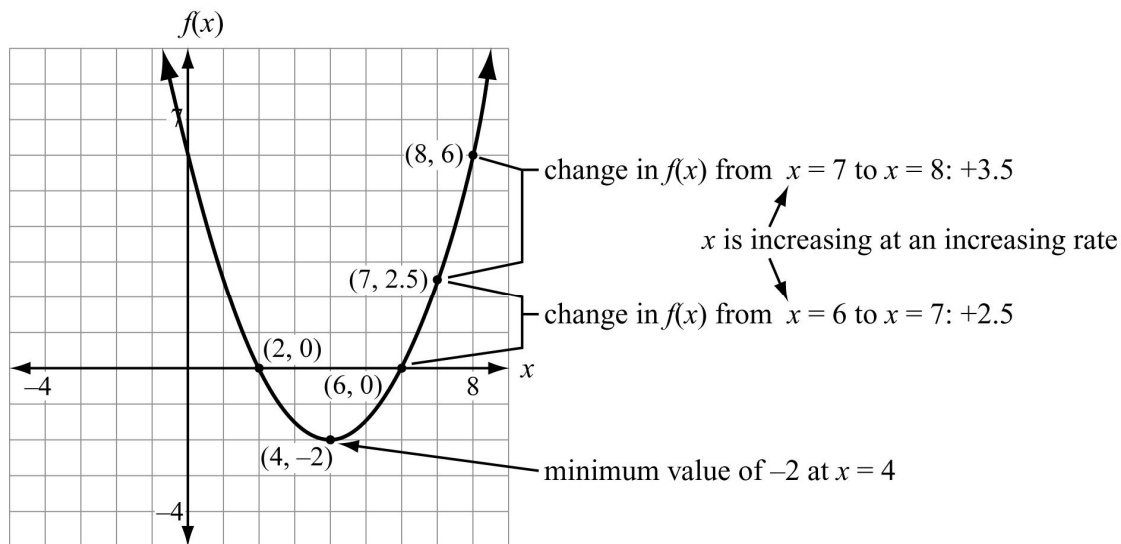
**Solution:**

We know from the function that the graph has a vertex at  $(2, 2)$ , it opens down, and has a width that is twice the width of  $f(x) = x^2$ . The sketch of the graph is shown below.



**Example:**

This graph shows a quadratic function. Identify the minimum value of the function, and show that the function is increasing at an increasing rate for  $x > 4$ . What is the value of  $a$  for this function?

**Solution:**

<b>Minimum value of <math>f(x)</math></b>	Occurs at low point of graph. At that point, the value of $x$ is 4 and the value of $f(x)$ is $-2$ .
<b>Behavior for <math>x &gt; 4</math></b>	At $x = 4$ , the value of $f(x)$ is $-2$ . At $x = 5$ , the value of $f(x)$ is $-1.5$ . At $x = 6$ , the value of $f(x)$ is $0$ . At $x = 7$ , the value of $f(x)$ is $2.5$ . At $x = 8$ , the value of $f(x)$ is $6$ . As $x$ increases, the value of $f(x)$ increases as well. The amount of increase is greater each time $x$ increases by 1, so $f(x)$ is increasing at an increasing rate for $x > 4$ .
<b>Value of <math>a</math></b>	The step-by-step increase in the value of $f(x)$ for each of the steps listed above is $0.5$ , $1.5$ , $2.5$ , and $3.5$ . Each of these steps is half of the corresponding step for the function $f(x) = x^2$ when starting from the vertex of the function. Since $f(x)$ has a minimum, $a$ is positive. Therefore, $a = \frac{1}{2}$ .



### Important Tip

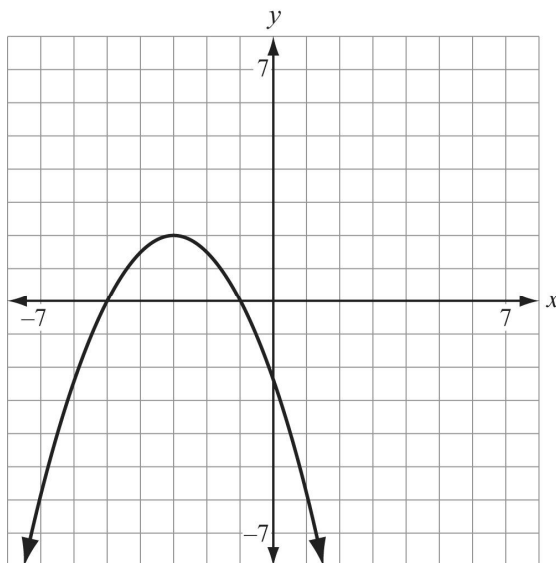
You can make a table of values to help determine the value of  $a$  if the graph is difficult to read precisely at the points that are 1 unit (in terms of  $x$ -value) from the vertex. For  $a = 1$ , the absolute value of the change in  $f(x)$  for each 1-unit change in  $x$  is the set of odd numbers: 1, 3,

5, 7, ... For  $a = \frac{1}{2}$ , the corresponding values are  $\frac{1}{2} \cdot 1 = \frac{1}{2}$ ,  $\frac{1}{2} \cdot 3 = 1\frac{1}{2}$ ,  $\frac{1}{2} \cdot 5 = 2\frac{1}{2}$ ,

$\frac{1}{2} \cdot 7 = 3\frac{1}{2}$ , ...

### REVIEW EXAMPLES

- 1) This graph represents a quadratic function.



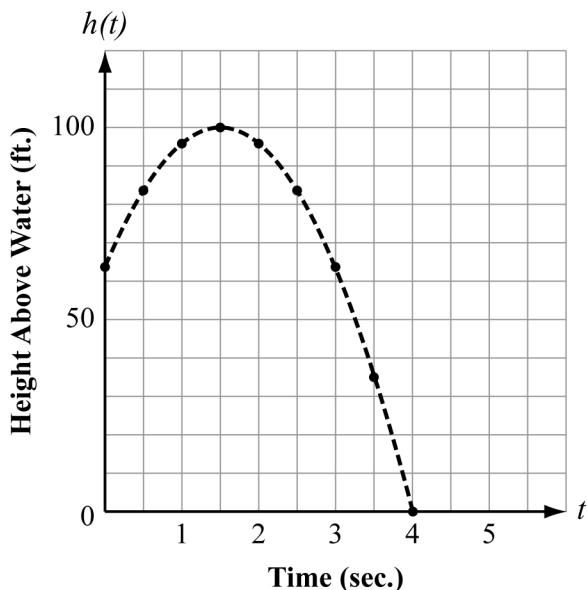
- Identify the location of the vertex.
- Write an equation to represent the axis of symmetry.
- Describe the horizontal and vertical shifts and the vertical stretch or shrink that could be applied to the graph of  $y = x^2$  to get this graph.
- Write an equation in the form  $y = a(x - h)^2 + k$  to represent this function.
- What are the zeros and the  $y$ -intercept of the function?
- What are the domain and range of the function?
- On what interval or intervals is the function increasing? On what interval or intervals is the function decreasing? For each interval, state whether the rate of increase or decrease is increasing or decreasing.

**Solution:**

- a.  $(-3, 2)$
- b.  $x = -3$
- c. Horizontal shift: 3 units left  
Vertical shift: 2 units up  
Stretch factor:  $-\frac{1}{2}$
- d.  $y = -\frac{1}{2}(x + 3)^2 + 2$
- e. Zeros:  $-5$  and  $-1$ ;  $y$ -intercept:  $-2.5$  or  $-\frac{5}{2}$
- f. Domain:  $\{-\infty < x < \infty\}$ , or  $(-\infty, \infty)$  Range:  $\{-\infty < y \leq 2\}$ , or  $(-\infty, 2]$
- g. The function is increasing on the interval  $\{-\infty < x < -3\}$ , or  $(-\infty, -3)$ ; the rate of increase is decreasing for the whole interval. The function is decreasing on the interval  $\{-3 < x < \infty\}$ , or  $(-3, \infty)$ ; the rate of decrease is increasing for the whole interval.

- 2) Philip is standing on a rock ledge that juts out over a lake. He tosses a rock straight up with a velocity of 48 feet per second. The rock leaves his hand at a point 64 feet above the surface of the lake. The rock travels upward and then falls into the lake.

This graph and table represent the height above the water,  $h(t)$ , as a function of the time,  $t$ , in seconds after Philip releases the rock.



Time ( $t$ )	Height ( $h(t)$ )
0	64
1	96
2	96
3	64
4	0

- What is the maximum height the rock reaches above the surface of the lake?
- After how many seconds does the rock hit the surface of the lake?
- Identify the vertex and the axis of symmetry of the graph.
- What does the  $y$ -intercept of the graph represent in this context? What does the  $x$ -intercept represent?
- For approximately how many seconds is the rock at least 80 feet above the surface of the lake?
- Write a function in the form  $h(t) = a(t-h)^2 + k$  that is represented by the graph.
- In this problem, what are the domain and range of  $h(t)$ ?
- Write the function from part f in standard form  $[h(t) = at^2 + bt + c]$ .
- Explain how the values of  $a$ ,  $b$ , and  $c$  in the function you wrote in part h relate to the problem situation.

**Solution:**

- 100 feet
- 4 seconds
- Vertex:  $(1.5, 100)$ ; Axis of symmetry:  $t = 1.5$  seconds
- The  $y$ -intercept represents the height from which Philip released the rock. The  $x$ -intercept represents the time,  $t$ , at which the rock hit the surface of the lake.
- About 2.2 seconds



f.  $h(t) = -16(t - 1.5)^2 + 100$

g. Domain:  $0 \leq t \leq 4$ , or  $[0, 4]$     Range:  $0 \leq h(t) \leq 100$ , or  $[0, 100]$

h.  $h(t) = -16t^2 + 48t + 64$

i.  $-16 \left( \frac{\text{ft.}}{\text{sec}^2} \right)$  is the force of gravity (negative means it's in the downward direction)

$48 \left( \frac{\text{ft.}}{\text{sec}} \right)$  is the initial velocity of the rock (positive means it's in the upward direction)

64 (ft.) is the initial height of the rock when it was released

**EOCT Practice Items**

1) The quadratic function  $f(x)$  has these characteristics:

- The vertex is located at  $(8, -2)$ .
- The range is  $-2 \leq f(x) < \infty$ .

Which function could be  $f(x)$ ?

- A.  $f(x) = \frac{1}{2}x^2 - 8x + 30$
- B.  $f(x) = \frac{1}{2}x^2 - 8x + 31$
- C.  $f(x) = -\frac{1}{2}x^2 + 8x - 34$
- D.  $f(x) = -\frac{1}{2}x^2 - 2x + 6$

[Key: A]

2) The vertex of the quadratic function  $g(x)$  is located at  $(4, 2)$ . An  $x$ -intercept of  $g(x)$  is located at  $(5, 0)$ . What is the  $y$ -intercept of  $g(x)$ ?

- A.  $(0, -30)$
- B.  $(0, -14)$
- C.  $(0, -4)$
- D.  $(0, 3)$

[Key: A]

## SOLUTIONS OF QUADRATIC FUNCTIONS



### KEY IDEAS

1. Factorable quadratic equations can be solved using this property of multiplication:

$$\text{if } ab = 0, \text{ then either } a = 0 \text{ or } b = 0$$

Thus, if  $(x-p)(x-q) = 0$ , then  $(x-p) = 0$  or  $(x-q) = 0$ ; i.e.,  $x = p$  or  $x = q$ .

2. Quadratic equations in the vertex form  $[0 = a(t-h)^2 - k]$  can be solved by adding  $k$  to each side and dividing by  $a$  to get  $\frac{k}{a} = (t-h)^2$ ; this means  $(t-h) = \pm\sqrt{\frac{k}{a}}$  and, therefore,
- $$t = h \pm \sqrt{\frac{k}{a}}.$$

3. The **quadratic formula** gives a general solution to **any** quadratic equation of the form  $0 = ax^2 + bx + c$ . The formula expresses the values of  $x$  that are solutions as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The value of the **discriminant**,  $b^2 - 4ac$ , determines the number and character of the solutions to a quadratic equation.

- If  $b^2 - 4ac > 0$ , the equation has **two** real solutions.
- If  $b^2 - 4ac = 0$ , the equation has **exactly one** real solution.
- If  $b^2 - 4ac < 0$ , the equation has **no** real solutions; there are two **complex** solutions.

Note that the number and character of the solutions to a quadratic equation correspond to the number of zeros of the graph of the quadratic function, as illustrated in Key Idea #6 on pages 98-99.

### REVIEW EXAMPLES

- 1) Consider the function  $f(x) = -\frac{1}{2}(x+3)^2 + 2$  from the previous set of review examples.
- Solve the equation  $f(x) = 0$ .
  - Write the function in standard form and use the quadratic formula to verify the solutions you found in part a.

**Solution:**

$$\text{a. } 0 = -\frac{1}{2}(x+3)^2 + 2$$

$$-2 = -\frac{1}{2}(x+3)^2$$

$$4 = (x+3)^2$$

$$(x+3) = 2 \text{ or } (x+3) = -2$$

$$x = -1 \text{ or } x = -5$$

$$\text{b. } 0 = -\frac{1}{2}x^2 - 3x - \frac{5}{2}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)}}{2\left(-\frac{1}{2}\right)}$$

$$x = \frac{3 \pm \sqrt{9 - 4\left(\frac{5}{4}\right)}}{-1}$$

$$x = \frac{3 \pm \sqrt{4}}{-1}$$

$$x = \frac{3+2}{-1} \text{ or } \frac{3-2}{-1} = -5 \text{ or } -1$$

Note that these solutions correspond to the zeros in the graph and that they are symmetrical about the axis of symmetry.

- 2) Consider the function from the previous review example in which Philip tossed a rock that landed in the lake. In standard form, the function is represented by  $h(t) = -16t^2 + 48t + 64$ .
- Solve the equation  $h(t) = 0$ .
  - Solve the equation  $h(t) = 80$ .

**Solution:**

$$\text{a. } 0 = -16t^2 + 48t + 64$$

This function has a common factor of  $-16$ :  $0 = -16(t^2 - 3t - 4)$

$$0 = (t^2 - 3t - 4)$$

$$0 = (t-4)(t+1)$$

$$t = -1 \text{ or } t = 4 \text{ (seconds)}$$

The solution  $t = 4$  corresponds with the solution found by reading the graph. Note that the solution  $t = -1$  has no meaning in this problem situation.



### ***Important Tip***

When finding solutions to a quadratic equation that models a real-world situation, always check to make sure the solutions have meaning in the problem situation.

$$\begin{aligned} \text{b. } 80 &= -16t^2 + 48t + 64 \\ 0 &= -16t^2 + 48t - 16 \\ 0 &= t^2 - 3t + 1 \quad (\text{divide both sides by the common factor of } -16) \end{aligned}$$

The expression  $t^2 - 3t + 1$  does not factor, so apply the quadratic formula

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{5}}{2}, \text{ so } x = \frac{3 + \sqrt{5}}{2} \approx 2.6 \text{ seconds or } x = \frac{3 - \sqrt{5}}{2} \approx 0.4 \text{ seconds.}$$

Note that these solutions both have meaning in this problem situation. Moreover, they can be used to find an exact answer  $[\sqrt{5} \text{ seconds}]$  to the question in part e of that review example.

- 3) Complete the table below by stating whether each quadratic equation is factorable, the value of the discriminant, the number and character of the solutions, and the real solutions.

Equation	Factorable? Y/N	Value of discriminant	Number and character of solutions	Real solutions
$2x^2 - 10x + 12 = 0$				
$x^2 - 10x - 12 = 0$				
$x^2 + 6x + 10 = 0$				
$x^2 + 6x + 9 = 0$				

**Solution:**

Equation	Factorable? Y/N	Value of discriminant	Number and character of solutions	Real solutions
$2x^2 - 10x + 12 = 0$	Y $2(x-3)(x-2) = 0$	4	2 real solutions	$x = 2$ $x = 3$
$x^2 - 10x - 12 = 0$	N	52	2 real solutions	$x = 5 + \sqrt{13}$ $x = 5 - \sqrt{13}$
$x^2 + 6x + 10 = 0$	N	-4	0 real solutions	none
$x^2 + 6x + 9 = 0$	Y $(x+3)^2 = 0$	0	1 real solution	$x = -3$



### **Important Tip**

When the value of the discriminant is a perfect square, the real solutions are rational.  
When the value of the discriminant is 0, there is exactly 1 real solution.

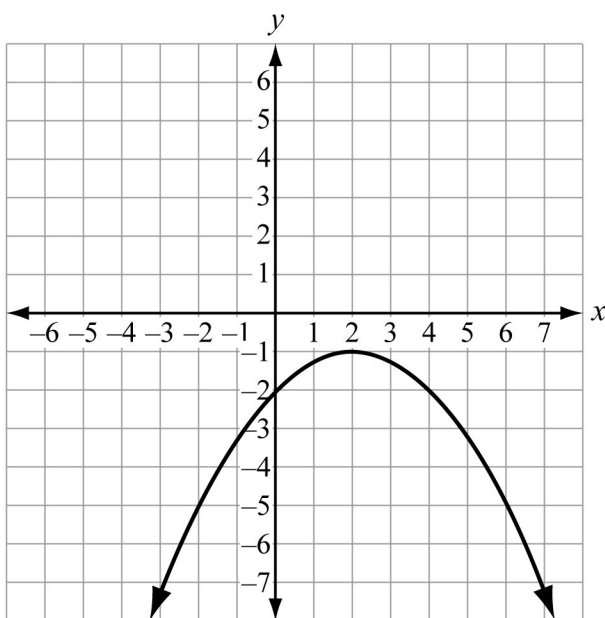
**EOCT Practice Items**

1) What are the solutions to the equation  $289 = \left(\frac{1}{3}x - 8\right)^2$  ?

- A.  $x = -27$  only
- B.  $x = 75$  only
- C.  $x = -27$  and  $x = 75$
- D.  $x = -75$  and  $x = 75$

[Key: C]

2) Use this graph of the function  $g(x)$  to answer the question.



Which statement about the real solutions of  $g(x)$  is true?

- A.  $g(x)$  has no real solutions.
- B.  $g(x)$  has exactly one real solution.
- C.  $g(x)$  has two real solutions that are rational.
- D.  $g(x)$  has two real solutions that are irrational.

[Key: A]

## IMAGINARY AND COMPLEX NUMBERS



### KEY IDEAS

1. The basis of the complex number system is the solution to the equation  $x^2 = -1$ . There is no real number that, when multiplied by itself, is equal to  $-1$ . This problem was solved by defining the imaginary number  $i$  such that  $i^2 = -1$ ; i.e.,  $i = \sqrt{-1}$ .
2. For quadratic equations in which the value of the discriminant is less than 0, the solutions have the form  $a \pm bi$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ . Such numbers are called *complex numbers*.

#### Example:

What are the solutions to the equation  $x^2 + 6x + 10 = 0$ ?

#### Solution:

We already know from review example #3 in the preceding section that this equation has no real solutions. When we apply the quadratic formula, we get

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(10)}}{2(1)}, \text{ which can be simplified to}$$

$$x = \frac{-6 \pm \sqrt{-4}}{2} = -3 + \frac{\sqrt{-4}}{2} \text{ or } -3 - \frac{\sqrt{-4}}{2}.$$

Applying the rules of arithmetic with respect to square roots and using the definition of  $i$ ,  $\sqrt{-4} = (\sqrt{4})(\sqrt{-1}) = 2i$ , allows the solutions to be simplified as

$$x = -3 + i \text{ or } x = -3 - i.$$



#### Important Tip

Complex solutions to quadratic equations will always have the form  $x = a + bi$  or  $x = a - bi$ . The numbers  $a + bi$  and  $a - bi$  are known as a *conjugate pair*.

3. Arithmetic operations on complex numbers follow the same rules that apply to arithmetic operations on real numbers and algebraic expressions. For example, addition, subtraction, and multiplication work as follows.



- $(a + bi) + (c + di) = (a + c) + (b + d)i$ ; that is, to add (or subtract) complex numbers, add the real parts and then add the imaginary parts.
- $(a + bi)(c + di) = ac + (bc)i + (ad)i + (bd)i^2$ . Since  $i^2 = -1$ , this expression can be simplified to  $(ac - bd) + (bc + ad)i$ .

Division of complex numbers makes use of the fact that ***the product of a complex number and its conjugate is a real number***. To divide complex numbers, multiply both the dividend and the divisor by the conjugate of the divisor, and then proceed according to the rules for multiplication and simplification of complex numbers.

$$\bullet \frac{(a + bi)}{(c + di)} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 - d^2i^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

4. Any complex number can be represented as a point on a coordinate plane in which the horizontal axis represents the real part and the vertical axis represents the imaginary part. The ***absolute value*** of a complex number is defined as the distance of the point  $(a, b)$  from the origin. After applying the Cartesian distance formula, the absolute value of  $a + bi$  is  $\sqrt{a^2 + b^2}$ .

### REVIEW EXAMPLES:

- 1) What are the values of  $i^5$ ,  $i^{17}$ , and  $i^{265}$ ?

#### Solution:

All three expressions are equivalent to  $i$ .

- 2) Simplify the following expressions:

- $2i + 5i$
- $2i \times 5i$
- $5i \div 2i$
- $i\sqrt{15} \times i\sqrt{40}$

#### Solution:

- $7i$
- $-10$
- $\frac{5}{2}$
- $-10\sqrt{6}$

3) Simplify the following expressions:

- $(3 + 2i) + (15 - 8i)$
- $(3 + 2i) - (15 - 8i)$
- $(3 + 2i) \times (15 - 8i)$
- $(15 - 8i) \div (3 + 2i)$
- $(3 + 2i)^3$
- $(3 + 2i) \times (3 - 2i)$

**Solution:**

- $18 - 6i$
- $-12 + 10i$
- $61 + 6i$
- $\frac{29}{13} - \frac{54}{13}i$
- $-9 + 46i$
- $13$

4) What is the absolute value of  $(15 - 8i)$ ?

**Solution:**

The absolute value is 17.



### **Important Tip**

As with all division problems, you can verify the result of division of complex numbers by multiplying the quotient by the divisor; the result should be the original dividend.

Using the complex numbers from review example 3d above,

$$\begin{aligned} \left(\frac{29}{13} - \frac{54}{13}i\right)(3 + 2i) &= \frac{87}{13} - \frac{162}{13}i + \frac{58}{13}i - \frac{108}{13}i^2 \\ &= \frac{195}{13} - \frac{104}{13}i = 15 - 8i \end{aligned}$$

**EOCT Practice Item**

Which expression is equivalent to  $\frac{12-5i}{2-i}$ ?

- A.  $\frac{19}{3} - \frac{22}{3}i$   
 B.  $\frac{29}{3} + \frac{2}{3}i$   
 C.  $\frac{19}{5} - \frac{22}{5}i$   
 D.  $\frac{29}{5} + \frac{2}{5}i$

[Key: D]

## ARITHMETIC SERIES AND SUMS OF FINITE ARITHMETIC SERIES



### KEY IDEAS

1. A **finite arithmetic series** is a series of numbers in which the difference between each term is constant and contains both a starting and ending term. For example, the series 1, 2, 3, 4, 5, ...,  $n$  is finite because the difference between each term and the next one is 1.
2. An arithmetic series can be defined in one of two ways: a **closed form** definition tells how to find the value of the  $n^{\text{th}}$  term. A **recursive** definition gives the first term and a formula for finding the  $(n+1)^{\text{th}}$  term, given the value of the  $n^{\text{th}}$  term. Using the series from Key Idea #1 and defining  $T_n$  as the  $n^{\text{th}}$  term, a closed form definition would be  $T_n = n$  and a recursive definition would be  $T_{n+1} = T_n + 1$ ;  $T_1 = 1$ .
3. Many different real-life situations can be solved by finding the partial sum of the finite arithmetic series 1, 2, 3, 4, ...,  $n$ . The partial sum of the series is  $1 + 2 + 3 + \dots + n$ . The sum can be found by pairing the terms  $(1+n) + (2+n-1) + (3+n-2) + \dots$ . The value of each term is equal to  $(n+1)$  and there are  $\frac{n}{2}$  of them; hence the sum is  $\frac{n(n+1)}{2}$ .

For an arithmetic series with a first term of  $a$  and a last term of  $b$  the sum is  $\frac{n(a+b)}{2}$ .

The partial sums of other arithmetic series can be seen as translations of this one.

4. The square numbers  $\{1, 4, 9, 16, 25, \dots\}$  are the partial sums of the arithmetic series of odd integers.

### REVIEW EXAMPLES

- 1) A university runs bus service to the following towns: Durham, Portsmouth, Lee, Dover, and Newington.
- How many routes are necessary if each route connects one of the towns with Durham?
  - How many routes are necessary if each route connects one of the towns with another, and there is a route that connects every pair of towns without having to go through Durham?
  - Suppose the university decides to add routes to two additional towns. How many additional routes will be necessary to meet the condition in part a? How many routes will be necessary to meet the conditions in part b?
  - Write a closed-form function,  $R(t)$ , that represents the number of necessary routes if there is service to  $t$  towns to meet the conditions in part b.

#### Solution:

- There are 4 towns other than Durham, so 4 routes are needed.
  - There are  $4 + 3 + 2 + 1 = 10$  routes needed.
  - 2 more routes are needed to meet the condition in part a;  $6 + 5 = 11$  more routes are needed to meet the conditions in part b.
  - $R(t) = \frac{t(t-1)}{2}$
- 2) Paul starts a new company that will build and sell guitars. His plan is to build 15 guitars the first month, 20 guitars the second month, and 25 guitars the third month. He plans to continue to increase the company's output by 5 guitars each month.
- What is the total number of guitars that Paul's company will build in the first year of production?
  - Paul plans to stabilize production when it reaches a level of 1000 guitars per month. How many years will it take to reach this level of production?
  - What is the total number of guitars that the company will have built when this point is reached?
  - Write a closed-form function,  $G(n)$ , that represents the total number of guitars that will have been built after  $n$  months of production.

**Solution:**

- a.  $15 + 20 + 25 + \dots + 70 = 12 \left( \frac{70 + 15}{2} \right) = 510$  guitars
- b. The month in which 1000 guitars will be built is the solution of  $10 + 5n = 1000$ . The solution is 198 months or  $16\frac{1}{2}$  years.
- c.  $15 + 20 + 25 + \dots + 1,000 = 198 \left( \frac{15 + 1,000}{2} \right) = 100,485$  guitars
- d.  $G(n) = n \left( \frac{15 + (10 + 5n)}{2} \right) = \frac{5}{2}n^2 + \frac{25}{2}n$

**EOCT Practice Items**

1) Which expression represents the sum of the first  $n$  multiples of 8?

- A.  $8n$   
B.  $8n^2$   
C.  $4n^2 + 4n$   
D.  $8n^2 + 8n$

[Key: C]

2) Alex started a business making bracelets. She sold 30 bracelets the first month. Her goal is to sell 6 more bracelets each month than she sold the previous month.

If Alex meets her goal, what is the total number of bracelets she will sell in the first 12 months?

- A. 378  
B. 426  
C. 498  
D. 756

[Key: D]

## **Unit 6: Putting the Pieces Together**

This unit investigates piecewise functions and their applications in a real-world context. Linear and quadratic regression is also explored. Students look at bivariate data to determine which type of curve best fits the data, including the extent to which a given line or curve is or is not a good fit for the data. Students use methods such as the median-median line and estimation to determine the equation of an appropriate line or curve to fit a given set of data, and use the equation to make predictions and analyze a real-world relationship.

**Note:** For the purposes of the Georgia End-of-Course Test, it is not possible to presume that all students will have access to technology that will apply the processes of linear and quadratic regression for curve fitting. As a result, determining the line or curve of best fit will not be directly assessed on the Georgia End-of-Course Test. Instead, the test will focus on assessing whether a given line or function is an appropriate model for a set of data.

### **KEY STANDARDS**

- MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.**
- Write absolute value functions as piecewise functions.
  - Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.
  - Solve absolute value equations and inequalities analytically, graphically, and by using appropriate technology.
- MM2D2. Students will determine an algebraic model to quantify the association between two quantitative variables.**
- Gather and plot data that can be modeled with linear and quadratic functions.
  - Examine the issues of curve fitting by finding good linear fits to data using simple methods such as the median-median line and “eyeballing.”
  - Understand and apply the processes of linear and quadratic regression for curve fitting using appropriate technology.
  - Investigate issues that arise when using data to explore the relationship between two variables, including confusion between correlation and causation.

## PIECEWISE FUNCTIONS



### KEY IDEAS

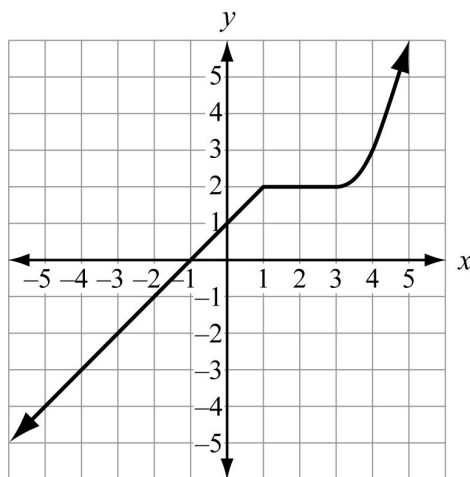
1. **Piecewise functions** are functions that can be represented by more than one equation, with each equation corresponding to a different part of the domain. Piecewise functions do not always have to be line segments. The “pieces” could be pieces of any type of graph. This type of function is often used to represent real-life problems.

#### Example:

$$f(x) = \begin{cases} x + 1, & \text{if } x < 1 \\ 2, & \text{if } 1 \leq x \leq 3 \\ (x - 3)^2 + 2, & \text{if } x > 3 \end{cases}$$

To graph this piecewise function, graph the function  $f(x) = x + 1$  for all values of  $x$  less than 1 or on the interval  $(-\infty, 1)$ ; graph the function  $f(x) = 2$  for all values of  $x$  from 1 to 3, including 1 and 3, or on the interval  $[1, 3]$ ; and graph the function  $f(x) = (x - 3)^2 + 2$  for all values of  $x$  greater than 3 or on the interval  $(3, \infty)$ .

This is the graph of the piecewise function in the example.



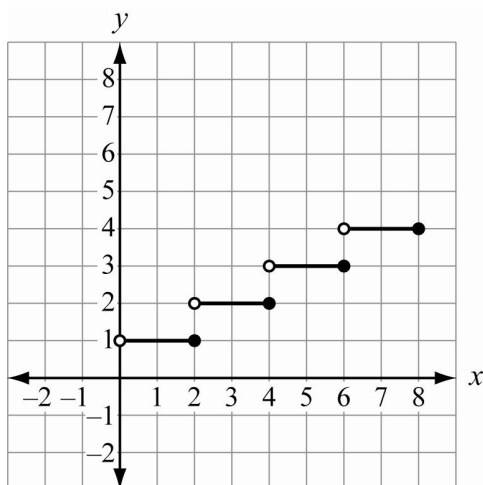
Notice that in this case the graph of the piecewise function is one continuous set of points because the individual graphs of each of the three pieces of the function connect. This is not true of all cases. The graph of a piecewise function may have a break or a gap where the pieces do not meet.

2. A **step function** is an example of a piecewise function.

**Example:**

$$f(x) = \begin{cases} 1, & \text{if } 0 < x \leq 2 \\ 2, & \text{if } 2 < x \leq 4 \\ 3, & \text{if } 4 < x \leq 6 \\ 4, & \text{if } 6 < x \leq 8 \end{cases}$$

This is a graph of the step function in the example.



3. Two particular kinds of step functions are called **ceiling functions** ( $f(x) = \lceil x \rceil$ ) and **floor functions** ( $f(x) = \lfloor x \rfloor$ ). In a ceiling function, all nonintegers are rounded up to the nearest integer. An example of a ceiling function is when a phone service company charges by the number of minutes used and always rounds up to the nearest integer of minutes. In a floor function, all nonintegers are rounded down to the nearest integer. The way we usually count our age is an example of a floor function since we round our age down to the nearest year and do not add a year to our age until we have passed our birthday. The **floor function** is the same thing as the **greatest integer function** which can be written as  $f(x) = \lceil x \rceil$  or  $f(x) = \lfloor x \rfloor$ .
4. An **absolute value function** is a special case of a piecewise function. The graph of an absolute value function makes a V-shape. The **vertex** of the graph is the point at the bottom of the V if the graph opens up, or the point at the top of the V if the graph opens down. The **axis of symmetry** of an absolute value function is the vertical line that passes through its vertex. For an absolute value function in the form  $f(x) = a|x - h| + k$ , the vertex is  $(h, k)$  and the line of symmetry is  $x = h$ .

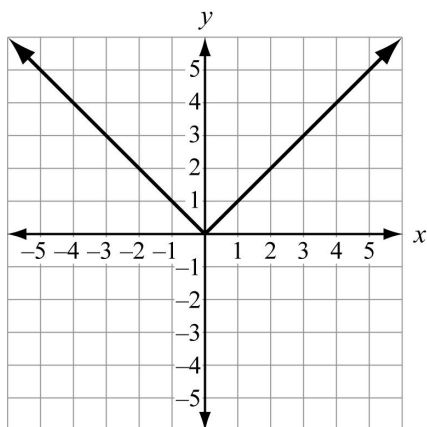


**Example:**

We usually write an absolute value function as  $f(x) = |x|$ , but since absolute value is a measure of distance and distance is always positive, it also can be written as follows:

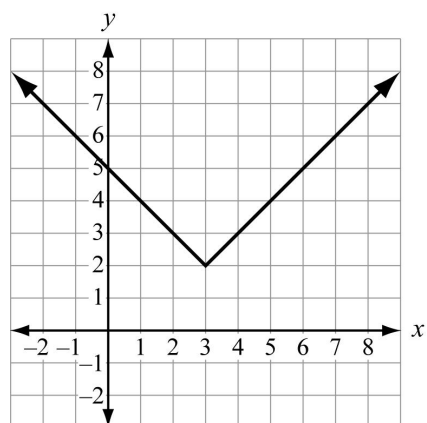
$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

This is the graph of  $f(x) = |x|$ .



5. If the graph of the absolute value function from Key Idea #4 is shifted to the right 3 and up 2, it can be represented by the function  $f(x) = |x - 3| + 2$ .

This is the graph of the absolute value function  $f(x) = |x - 3| + 2$ .

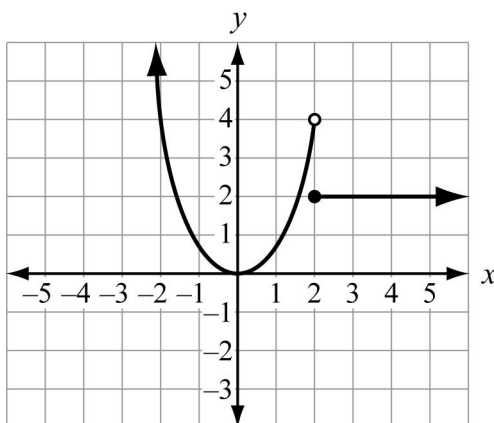


6. Remember that the **domain** of a function is the set of input numbers, and the **range** is the set of output numbers. In a piecewise function, the input number determines which equation to use to find the output number.

7. A function may have a **maximum** (highest point) and/or a **minimum** (lowest point) or neither. In an absolute value function, the vertex is at the maximum if the V opens down and at the minimum if the V opens up.
8. A **point of discontinuity** is a point where there is a break or a gap in the graph. A graph is said to be discontinuous when there is a break or a gap in it.

**Example:**

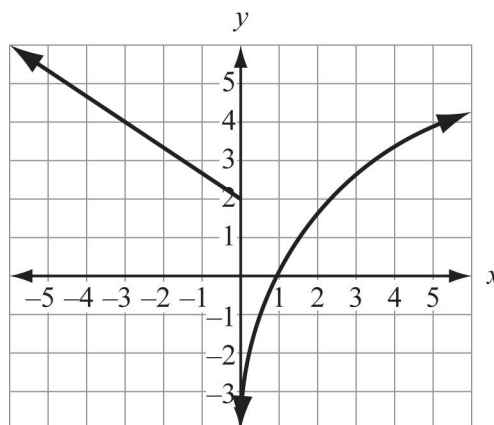
This is a graph of a piecewise function that is also a discontinuous function. It is discontinuous at  $x = 2$ .



The open dot means that the point  $(2, 4)$  does not belong to the graph of the first piece. The closed dot means that the point  $(2, 2)$  belongs to the graph of the second piece. Notice that the function still passes the vertical line test. The vertical line  $x = 2$  only passes through one point,  $(2, 2)$ .

**Example:**

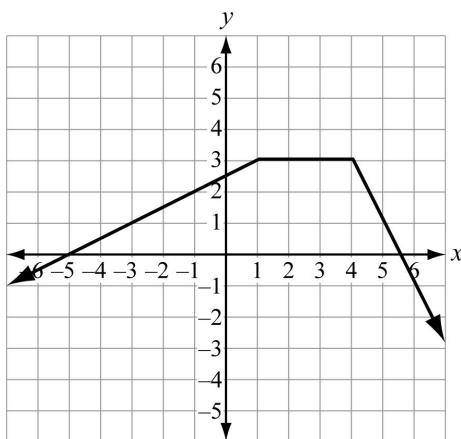
This is another graph of a piecewise function that is also a discontinuous function. It is discontinuous at  $x = 0$ . In this case, the function is **undefined** at  $x = 0$ .



9. An interval for which a function is **constant** is the interval where the graph does not rise or fall. An interval for which a function **increases** is one where the function rises, while an interval for which a function **decreases** is one where the function falls. In other words, when the domain or the  $x$ -values of a function increases and the range or  $y$ -values increase, a function is said to increase. When the domain increases and the range decreases, a function is said to decrease.

**Example:**

This is a graph of a piecewise function that increases on the interval  $x \leq 1$ , decreases on the interval  $x \geq 4$ , and is constant on the interval  $1 \leq x \leq 4$ .



10. One way to solve an **absolute value equation** is to rewrite it as two linear equations and solve each equation. If the absolute value equation is complex, the first step is to isolate the expression containing the absolute value symbols. Always check each solution to be sure it works in the original equation.

**Example:**

In this case, the absolute value of the expression  $3x - 4$  is given as 11. That means the value of  $3x - 4$  must have a distance from zero of 11 units, which can be in a positive or negative direction. When we rewrite it as two linear equations, we set the expression equal to 11 and then set it equal to  $-11$ . It is written as a disjunction using the word “or.”

$$|3x - 4| = 11$$

$$3x - 4 = 11 \quad \text{or} \quad 3x - 4 = -11$$

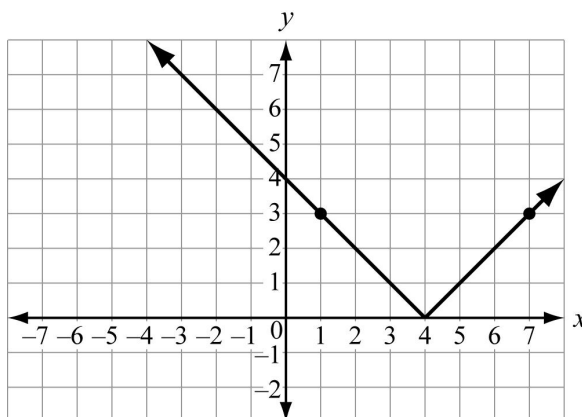
$$3x = 15 \quad \text{or} \quad 3x = -7$$

$$x = 5 \quad \text{or} \quad x = -\frac{7}{3}$$

11. Another way to solve an absolute value equation is to graph it on a coordinate grid.

**Example:**

To solve  $|x - 4| = 3$  by graphing, first rewrite it as  $f(x) = |x - 4|$ . Graph the function. The solution would be the  $x$ -values when the  $y$ -values are 3.



By looking at the graph, you can see that the solutions or the  $x$ -values when the  $y$ -values are 3 would be  $x = 1$  or  $x = 7$ .

12. One way to solve an **absolute value inequality** in the form  $|ax + b| < c$  or  $|ax + b| \leq c$  is to rewrite it as a compound inequality.

**Example:**

In this case, the absolute value of the expression  $2x + 5$  is given as being less than 13. This means that the value of  $2x + 5$  must have a distance from zero of less than 13 units, which can be in a positive direction or a negative direction. Therefore we write it as a conjunction and solve it.

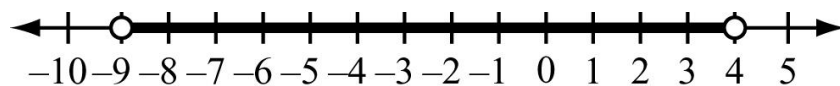
$$|2x + 5| < 13$$

$$-13 < 2x + 5 < 13$$

$$-18 < 2x < 8$$

$$-9 < x < 4$$

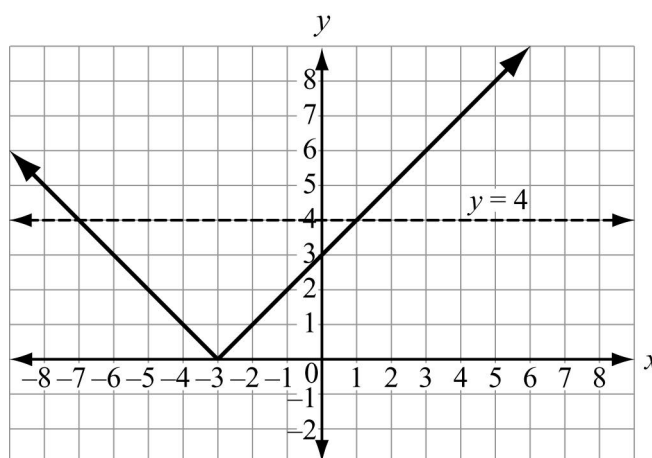
The solution can be any real number between  $-9$  and positive  $4$ . This is the graph of the absolute value inequality on a number line.



13. Another way to solve an absolute value inequality in the form  $|ax + b| < c$  or  $|ax + b| \leq c$  is to graph it on a coordinate grid.

**Example:**

To solve  $|x + 3| < 4$  by graphing, first rewrite it as  $f(x) = |x + 3|$ . Graph the function. The solution would be all of the  $x$ -values when the  $y$ -values are less than 4.



A dotted line is drawn across the graph at  $y = 4$ . All of the  $x$ -values on the graph that are below or less than the dotted line are solutions. The solution is the set of all real numbers greater than  $-7$  and less than  $4$ , or  $-7 < x < 4$ .

14. One way to solve an **absolute value inequality** in the form  $|ax + b| > c$  or  $|ax + b| \geq c$  is to rewrite it as a disjunction and solve each part of the disjunction.

**Example:**

In this case, the absolute value of the expression  $2x + 5$  is given as being greater than 13. This means that the value of  $2x + 5$  must have a distance from zero of more than 13 units, which can be in a positive direction or a negative direction. There we write it as a disjunction and solve it as shown.

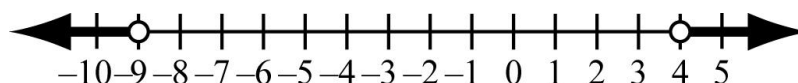
$$|2x + 5| > 13$$

$$2x + 5 < -13 \quad \text{or} \quad 2x + 5 > 13$$

$$2x < -18 \quad \text{or} \quad 2x > 8$$

$$x < -9 \quad \text{or} \quad x > 4$$

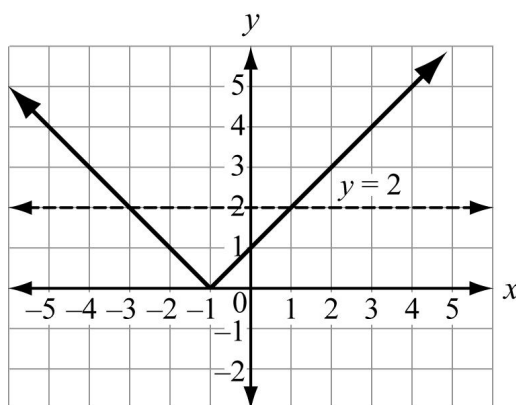
This is the graph of the absolute value inequality on a number line.



15. Another way to solve an inequality in the form  $|ax + b| > c$  or  $|ax + b| \geq c$  is to graph it on a coordinate grid.

**Example:**

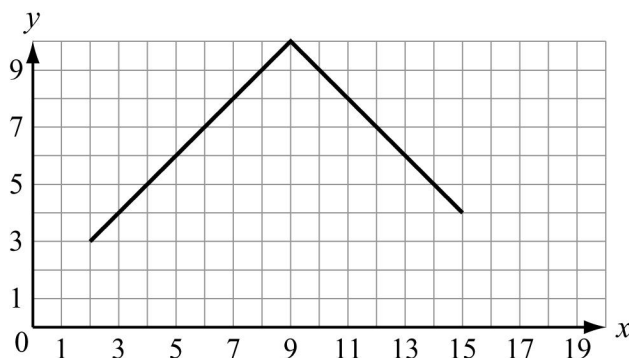
To solve  $|x + 1| > 2$  by graphing, first rewrite it as  $f(x) = |x + 1|$ . Graph the function. The solution would be all of the  $x$ -values when the  $y$ -values are greater than 2.



A dotted line is drawn across the graph at  $y = 2$ . All of the  $x$ -values on the graph that are above or greater than the dotted line are solutions. The solution is the set of all real numbers greater than 1 or less than  $-3$ , or the solution can be written as  $x > 1$  or  $x < -3$ .

**REVIEW EXAMPLES**

- 1) A coordinate grid represents a rectangular pool table. A ball is on a pool table at the point  $(2, 3)$ . The ball is rolled so that it hits the side of the pool table at the point  $(9, 10)$ . Then it rolls toward the other side, as shown in this diagram.



- Write a piecewise function that can represent the path of the ball.
- If the ball continues to roll, at what point will it hit the other side of the pool table?
- What do the  $x$ -value and the  $y$ -value represent?

**Solution:**

- a. To find a function that can represent the path of the ball, notice that the path of the ball forms a V. That means it can be represented by an absolute value function. Identify the vertex as point (9, 10). The slope (rise over run) is 1, and since the V opens down,  $a$  or the slope of the function is  $-1$ . Substitute this information into the absolute value function formula  $f(x) = a|x - h| + k$ . The function is  $f(x) = -|x - 9| + 10$ .

This can be written as a piecewise function so that it represents only the part of the path shown in the diagram that the ball actually traveled.

$$f(x) = \begin{cases} x + 1, & \text{if } 2 \leq x < 9 \\ -x + 19, & \text{if } 9 \leq x \leq 15 \end{cases}$$

To write the piecewise function from  $f(x) = -|x - 9| + 10$ , follow these steps:

If  $x - 9 \geq 0$ ,

then  $|x - 9| = x - 9$

$$\text{so } f(x) = -(x - 9) + 10$$

$$f(x) = -x + 9 + 10$$

$$f(x) = -x + 19$$

If  $x - 9 < 0$ ,

then  $|x - 9| = -(x - 9)$

$$\text{so } f(x) = -(-(x - 9)) + 10$$

$$f(x) = -(-x + 9) + 10$$

$$f(x) = x - 9 + 10$$

$$f(x) = x + 1$$

The interval is determined by looking at the graph for the actual path of the ball.

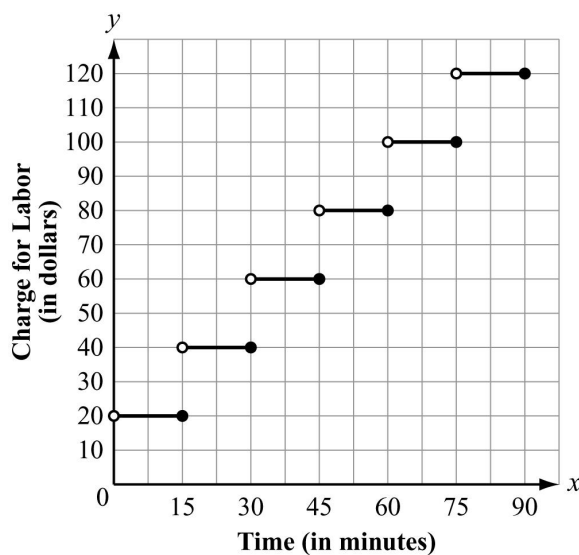
For  $f(x) = x + 1$ , the interval is  $2 \leq x < 9$ , and for  $f(x) = -x + 19$ , the interval is  $9 \leq x \leq 15$ .

- b. The ball will hit the other side at (19, 0).
- c. The  $x$ -value represents the distance to the right of the diagram of the pool table, and the  $y$ -value represents the distance up from the bottom of the diagram of the pool table.
- 2) A computer repair person charges \$80 per hour for labor. She charges her labor in increments of 15 minutes. For example, if she works for 39 minutes, she rounds up to 45 minutes and charges \$60.
- a. Write a function to represent the amount the repair person charges up to and including 90 minutes of labor.
- b. Graph the function from part a. Let  $x$  represent the number of minutes of labor charged.

**Solution:**

$$a. \quad f(x) = \begin{cases} 20, & \text{if } 0 < x \leq 15 \\ 40, & \text{if } 15 < x \leq 30 \\ 60, & \text{if } 30 < x \leq 45 \\ 80, & \text{if } 45 < x \leq 60 \\ 100, & \text{if } 60 < x \leq 75 \\ 120, & \text{if } 75 < x \leq 90 \end{cases}$$

b.

**EOCT Practice Items**1) Which function is equivalent to  $f(x) = 2|x + 2| + 1$ ?

- A.  $f(x) = \begin{cases} 2x + 5, & \text{if } x \geq -2 \\ -2x - 3, & \text{if } x < -2 \end{cases}$
- B.  $f(x) = \begin{cases} 2x + 5, & \text{if } x \geq 1 \\ -2x - 3, & \text{if } x < 1 \end{cases}$
- C.  $f(x) = \begin{cases} -2x - 5, & \text{if } x \geq -2 \\ 2x + 3, & \text{if } x < -2 \end{cases}$
- D.  $f(x) = \begin{cases} -2x - 5, & \text{if } x \geq 1 \\ 2x + 3, & \text{if } x < 1 \end{cases}$

[Key: A]

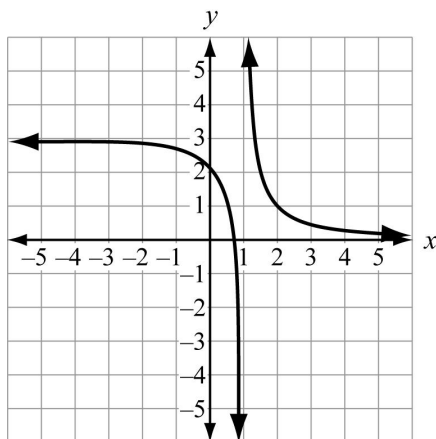


- 2) What is the function that results from multiplying  $f(x) = |x|$  by  $-1$  and shifting it 2 units to the right?

- A.  $f(x) = \begin{cases} -x - 2, & \text{if } x \leq 2 \\ x + 2, & \text{if } x > 2 \end{cases}$
- B.  $f(x) = \begin{cases} -x - 2, & \text{if } x \leq 0 \\ x + 2, & \text{if } x > 0 \end{cases}$
- C.  $f(x) = \begin{cases} x - 2, & \text{if } x \leq 2 \\ -x + 2, & \text{if } x > 2 \end{cases}$
- D.  $f(x) = \begin{cases} x - 2, & \text{if } x \leq 0 \\ -x + 2, & \text{if } x > 0 \end{cases}$

[Key: C]

- 3) This graph shows the two parts of a piecewise function.



For what value of  $x$  is the function NOT defined?

- A.  $-1$
- B.  $0$
- C.  $1$
- D.  $2$

[Key: C]

## DETERMINING AN APPROPRIATE ALGEBRAIC MODEL



### KEY IDEAS

1. **Bivariate data** are data that involve two variables that may be related to each other. The data can be presented as ordered pairs and in any way that ordered pairs can be presented: as a set of ordered pairs, as a table of values, or as a graph on the coordinate plane.
2. Bivariate data may have an underlying relationship that can be modeled by a mathematical function. For the purposes of this unit, we will consider models that are either **linear** or **quadratic** functions.

#### Example:

Evan is researching if there is a relationship between study time and mean test scores. He recorded the mean study time per test and the mean test score for students in four different courses. This is the data for Course 1.

Course 1

Mean Study Time (hours)	Mean Test Score
0.5	65
1.0	71
1.5	75
2.0	82
2.5	84
3.0	90
3.5	95

Notice that, for these data, as the mean study time increases, the mean test score increases. It is important to consider the **rate of increase** when deciding which algebraic model to use. In this case, the mean test score increases by approximately 5 points for each 0.5-hour increase in mean study time. When the rate of increase is close to constant as it is here, the best model is most likely a linear function.

This table shows Evan's data for Course 2.

### Course 2

Mean Study Time (hours)	Mean Test Score
0.5	65
1.0	66
1.5	68
2.0	73
2.5	79
3.0	87
3.5	98

In these data as well, the mean test score increases as the mean study time increases. However, the rate of increase is not constant. The differences between each successive mean test score are 1, 2, 5, 6, 8, and 11. The *second differences* are 1, 3, 1, 2, and 3. Since the second differences are fairly close to constant, it is likely that a quadratic function would be a good model for the data for Course 2.

This table shows Evan's data for Course 3.

### Course 3

Mean Study Time (hours)	Mean Test Score
0.5	65
1.0	74
1.5	80
2.0	84
2.5	86
3.0	85
3.5	81

In these data, the mean test score increases as the mean study time increases, but after a maximum point is reached the mean test score appears to decrease. The rate of increase decreases as it approaches the maximum, then, after the maximum is met, the rate of decrease increases. This is the characteristic behavior of a quadratic function that has a maximum, so it is likely that a quadratic function would be a good model for the data for Course 3 as well. However, since the function appears to have a maximum, the value of the coefficient of  $x^2$  would be negative rather than positive as it would be for Course 2.

This table shows Evan's data for Course 4.

**Course 4**

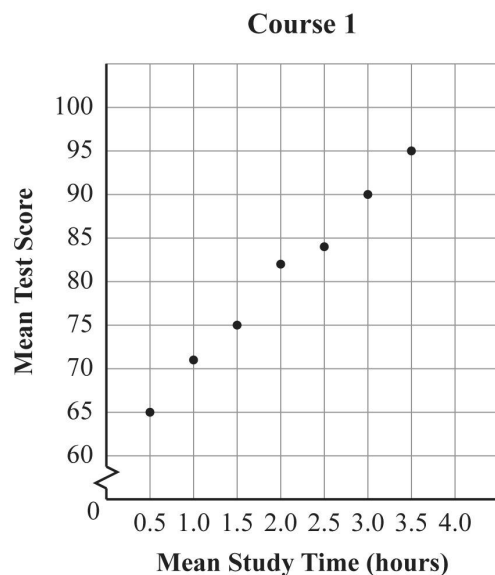
Mean Study Time (hours)	Mean Test Score
0.5	74
1.0	91
1.5	82
2.0	92
2.5	70
3.0	76
3.5	90

In these data, as the mean study time increases, there is no consistent pattern in the mean test score. As a result, there does not appear to be any clear relationship between mean study time and mean test score for this particular course.

Often, patterns in bivariate data are more easily seen when the data is plotted on a coordinate grid.

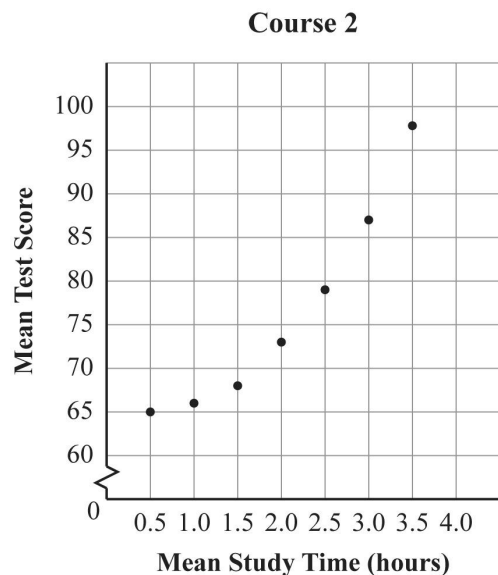
**Example:**

This graph shows Evan's data for Course 1.



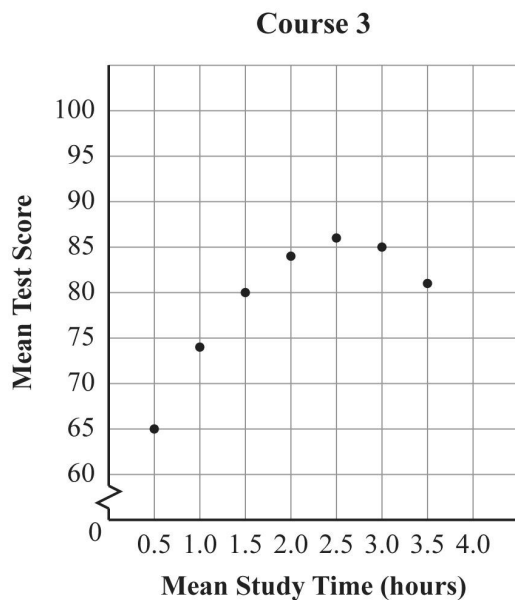
In this graph, the data points are all very close to being on the same line. This is further confirmation that a linear model is appropriate for this course.

This graph shows Evan's data for Course 2.



In this graph, the data points appear to lie on a curve, rather than on a line, with a rate of increase that increases as the value of  $x$  increases. It appears that a quadratic or exponential model may be more appropriate than a linear model for these data.

This graph shows Evan's data for Course 3.



In this graph, the data points appear to lie on a curve, rather than on a line, as well. However, in this case, the curve appears to have a maximum somewhere around  $x = 2.5$  hours. As a result, a quadratic model appears to be a good fit for this set of data.

## LINEAR MODELS



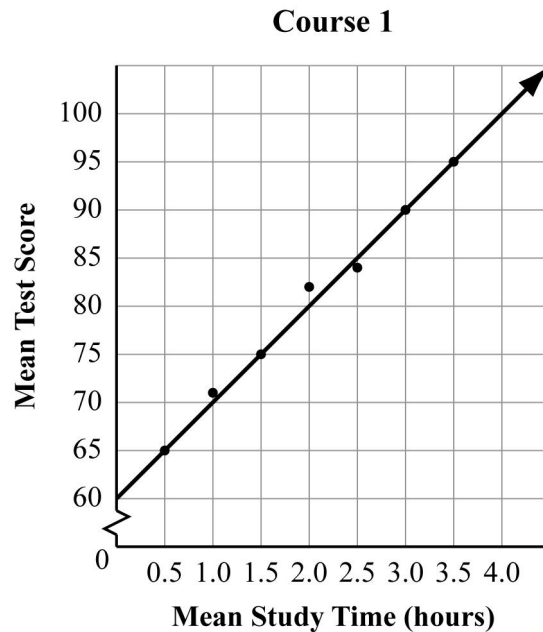
### KEY IDEAS

1. The extent to which a linear model does or does not fit a set of data is known as **correlation**. The amount of correlation is found by determining the distance of the data points from the **line of best fit**, which is the line that minimizes that distance. Correlation is usually expressed as a value between 1 and  $-1$ , with 1 representing perfect correlation, 0 representing no correlation, and  $-1$  representing perfectly negative correlation. In the context of the data relating study time and quiz grades, the correlation would be negative if the mean test score decreased as study time increased.
2. The **median-median line** is a way to estimate a line of best fit that involves relatively simple calculations. Since it involves using medians, it is also somewhat resistant to the effect of outliers in the data. To calculate a median-median line, order the data from the least to the greatest value of the  $x$ -coordinate. Order data points that have the same value of  $x$  from the least to the greatest value of the  $y$ -coordinate. Next, use this ordering to divide the data into three equal groups. Find the median  $x$ -coordinate value for each of the three groups (low values, middle values, high values). Then find the median of the  $y$ -values associated with each of these data points. Note that the median values of  $x$  and  $y$  may or may not be associated with the same data point. Find the equation of the line containing the two outside points (the points from the low- $x$  and the high- $x$  value sets). Then adjust the position of the line by moving it  $\frac{1}{3}$  of the way toward the middle point.

**Note:** Each item on the **GPS Algebra EOCT** that asks students to find the median-median line requires this method of calculation. Graphing calculators are currently not permitted for use during the **GPS Algebra EOCT**. Students should become familiar with this method as preparation for the assessment.

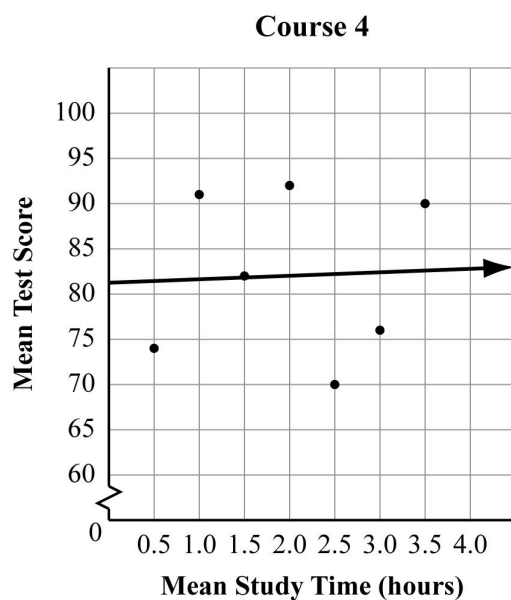
**Example:**

This graph shows Evan's data for Course 1 with the line of best fit added. The equation of the line is  $y = 10x + 60$ .



Notice that four of the seven data points are on the line. This represents a very strong correlation. Since the slope of the line is positive, the correlation is positive.

This graph shows Evan's data for Course 4 with the line of best fit added. The equation of the line is  $y = 0.4x + 81.3$ .



Although a line of best fit can be calculated for this set of data, notice that most of the data points are not very close to the line. In this case, although there is some correlation between study time and test scores, the amount of correlation is very small.

### REVIEW EXAMPLE

- 1) Tina collected the height in inches and the shoe size of a random sample of nine boys in her class. Her data is shown in this list.

$$\{(62, 6.5) (64, 7) (64, 8) (65, 8) (67, 8.5) (68, 10) (69, 9.5) (69, 11.5) (73, 11)\}$$

In each ordered pair, the  $x$ -coordinate is the height in inches and the  $y$ -coordinate is the shoe size.

Find the equation of the median-median line for Tina's data.

### Solution:

Tina's data is already in order from the least to the greatest  $x$ -value. The lowest set of  $x$ -values is  $\{62, 64, 64\}$ . The middle set is  $\{65, 67, 68\}$ . The highest set is  $\{69, 69, 73\}$ . The median values of  $x$  for these sets are 64, 67, and 69. For the lowest set, there are two data points that contain the value 64. The median point is  $(64, 7)$  because it is the middle point in the order. Similarly, the median point in the highest set is  $(69, 11)$ . Note that the  $y$ -value is *not* the median value of  $y$  for that set of three points; it is the  $y$ -value of the middle point.



Next, find the value of the line that contains the points (64, 7) and (69, 11).

The slope is  $\frac{(11-7)}{(69-64)} = \frac{4}{5} = 0.8$ . The  $y$ -intercept,  $b$ , is

$$7 = 0.8(64) + b$$

$$7 = 51.2 + b$$

$$-44.2 = b$$

The equation of the line containing the median points of the lowest and highest sets is  $y = 0.8x - 44.2$ . To adjust toward the median point of the middle set, plug the  $x$ -value of that point into the equation we just found.

$$y = 0.8(67) - 44.2$$

$$y = 53.6 - 44.2$$

$$y = 9.4$$

So the expected value of  $y$  from that equation is 9.4. The  $y$ -value of the median point is 8.5. This is 0.9 less than the expected value of  $y$ . One-third of that distance is 0.3, so subtract 0.3 from the  $y$ -intercept in the equation above to find the equation of the median-median line.

$$y = 0.8x - (44.2 - 0.3)$$

$$y = 0.8x - 43.9$$

## QUADRATIC MODELS



### KEY IDEA

Data such as that which Evan collected for Course 2 and Course 3 can be modeled by a quadratic function. As with data that can be modeled by a linear function, the idea is to choose a function that will minimize the distances from each data point to the graph of the function.

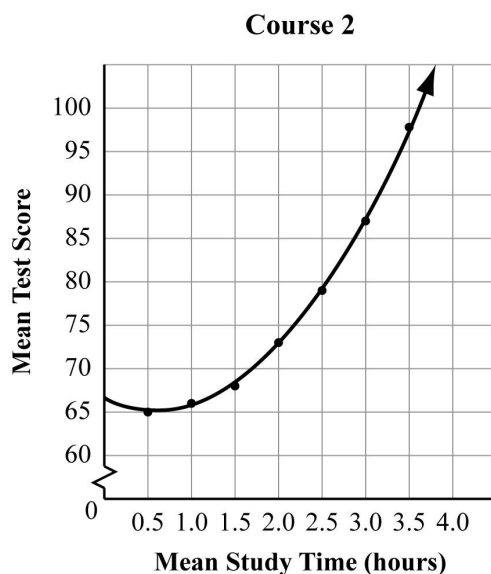


### Important Tip

Even when a quadratic function appears to be a good model for a particular set of data, it is important to consider the context when making predictions based on that model. For example, in the context of study time vs. test scores, a quadratic function may not be an appropriate model for values outside the ones in Evan's data. According to the model for the Course 2 data, as mean study time increases beyond 3.5 hours, mean test scores should continue to increase at an increasing rate. However, in most cases there is a maximum possible test score.

### Example:

This graph shows Evan's data for Course 2 with the quadratic regression curve added. The equation of the curve is  $y = 3.9x^2 - 4.8x + 66.6$ .



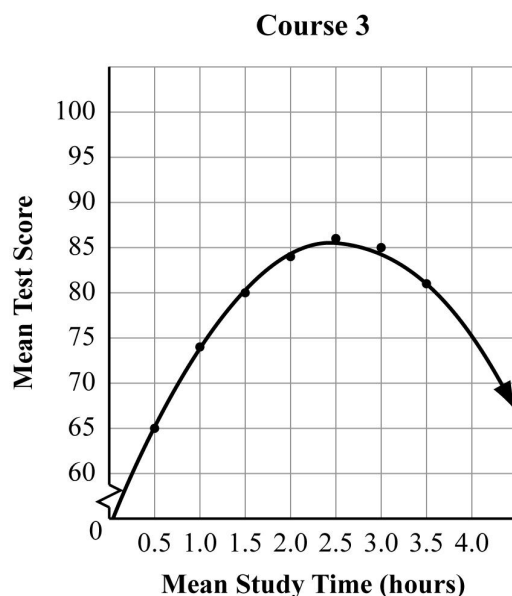
Notice that all seven data points are very close to being on the curve, so this function is a very good model for these data. However, it may not be appropriate to use this model to predict the mean test score associated with a mean study time of six hours. Substituting a

value of  $x = 6$  into this model gives this predicted test score:

$$y = 3.9(6)^2 - 4.8(6) + 66.6 = 140.4 - 28.8 + 66.6 = 178.2$$

If the maximum possible test score is 100, this model is clearly inappropriate for  $x$ -values greater than 3.5.

This graph shows Evan's data for Course 3 with the quadratic regression curve added. The equation of the curve is  $y = -4.95x^2 + 25.2x + 53.6$ . Notice that the negative coefficient of the  $x^2$  term indicates that this is a quadratic function with a graph that opens down, which means that it has a maximum value.



Notice again that all seven data points are very close to being on the curve, so this function is a very good model for these data. As in the previous model, it may not be appropriate to use this model to predict the mean test score associated with a mean study time of six hours. Substituting a value of  $x = 6$  into this model gives this predicted test score:  $y = -4.95(6)^2 + 25.2(6) + 53.6 = -178.2 + 151.2 + 53.6 = 26.6$

In this case, while a mean test score of about 27 is possible, it seems unlikely that students who studied for that long would have such low test scores.

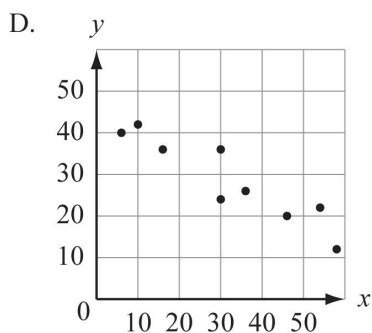
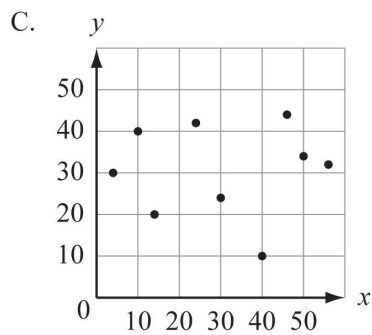
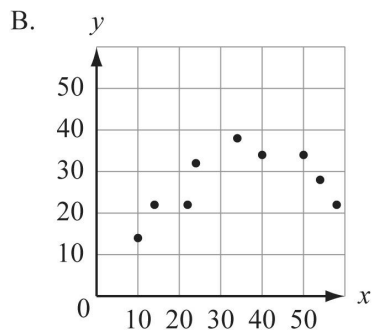
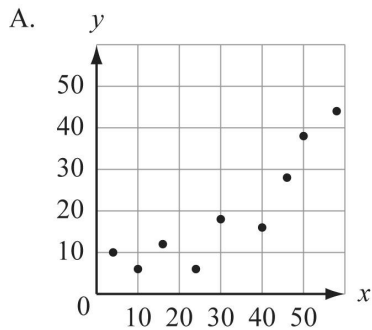


### **Important Tip**

When deciding whether or not a quadratic function is an appropriate model for a data set, you can make a table of values to show the  $y$ -values that the function predicts for the values of  $x$  in the data set and compare those  $y$ -values to the actual values from the data. This is also a good way to decide whether or not the quadratic model is appropriate for values outside the domain of the given data.

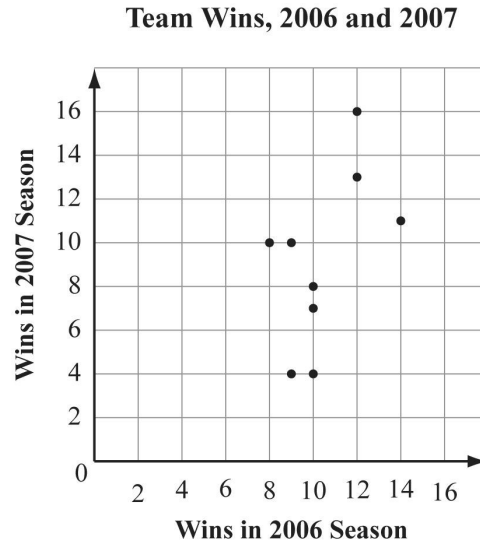
**EOCT Practice Items**

1) For which graph of a set of data is a linear function the best model?



[Key: D]

- 2) This graph plots the number of wins in the 2006 and 2007 seasons for a sample of professional football teams.

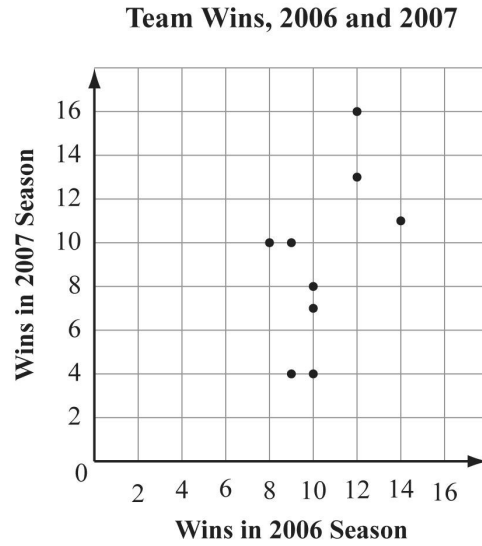


What is the equation of the median-median line for these data?

- A.  $y = x + 1$
- B.  $y = x - \frac{1}{3}$
- C.  $y = 4x - 32\frac{1}{3}$
- D.  $y = 4x - 32$

[Key: B]

- 3) This graph plots the number of wins in the 2006 and 2007 seasons for a sample of professional football teams.

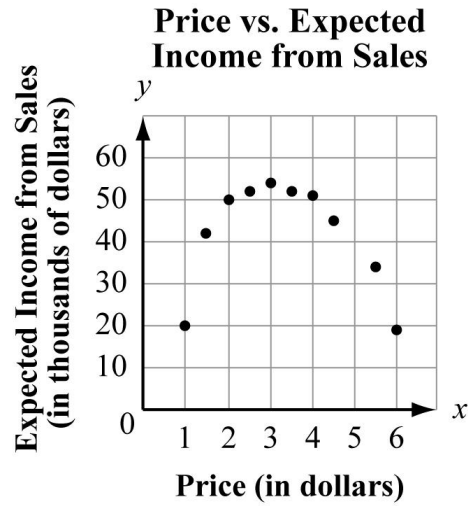


The linear regression model for these data is  $y = 1.10x - 2.29$ . Based on this model, what is the predicted number of 2007 wins for a team that won 5 games in 2006?

- A. 3
- B. 4
- C. 5
- D. 6

[Key: A]

- 4) This graph shows the expected income from sales vs. price per issue for a new magazine.



Which equation models these data?

- A.  $y = -5.1x^2 + 34.4x - 3.0$   
 B.  $y = 5.1x^2 - 34.4x + 3.0$   
 C.  $y = -34.4x^2 + 5.1x - 3.0$   
 D.  $y = 34.4x^2 - 5.1x + 3.0$

[Key: A]

5) This graph shows the price of a new audio-visual component over time.

**Price of New Audio-Visual  
Component Over Time**

Year	Median Price (in dollars)
0	500
2	425
4	360
6	280
8	200

**Which is the best explanation for why a linear model may not be appropriate for these data?**

- A. The median price decreases over time.
- B. The value of the median price cannot be negative.
- C. The value of the median price cannot continue to decrease.
- D. A quadratic model would be a better fit because the rate of decrease is decreasing.

[Key: B]



## Appendix A

### EOCT Sample Overall Study Plan Sheet

Here is a sample of what an OVERALL study plan might look like. You can use the Blank Overall Study Plan Sheet in Appendix B or create your own.

#### Materials/Resources I May Need When I Study:

(You can look back at page 6 for ideas.)

1. *This study guide*
2. *Pens/pencils*
3. *Highlighter*
4. *Notebook*
5. *Dictionary*
6. *Calculator*
7. *Mathematics textbook*

#### Possible Study Locations:

- First choice: *The library*
- Second choice: *My room*
- Third choice: *My mom's office*

#### Overall Study Goals:

1. *Read and work through the entire study guide.*
2. *Answer the sample questions and study the answers.*
3. *Do additional reading in a mathematics textbook.*

Number of Weeks I Will Study: *6 weeks*

Number of Days a Week I Will Study: *5 days a week*

#### Best Study Times for Me:

- Weekdays: *7:00 p.m. – 9:00 p.m.*
- Saturday: *9:00 a.m. – 11:00 a.m.*
- Sunday: *2:00 p.m. – 4:00 p.m.*

## Appendix B

### Blank Overall Study Plan Sheet

**Materials/Resources I May Need When I Study:**  
(You can look back at page 6 for ideas.)

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_
4. \_\_\_\_\_
5. \_\_\_\_\_
6. \_\_\_\_\_

**Possible Study Locations:**

- First choice: \_\_\_\_\_
- Second choice \_\_\_\_\_
- Third choice \_\_\_\_\_

**Overall Study Goals:**

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_
4. \_\_\_\_\_
5. \_\_\_\_\_

**Number of Weeks I Will Study:** \_\_\_\_\_

**Number of Days a Week I Will Study:** \_\_\_\_\_

**Best Study Times for Me:**

- Weekdays: \_\_\_\_\_
- Saturday: \_\_\_\_\_
- Sunday: \_\_\_\_\_

## Appendix C

### EOCT Sample Daily Study Plan Sheet

Here is a sample of what a DAILY study plan might look like. You can use the Blank Daily Study Plan Sheet in Appendix D or create your own.

#### Materials I May Need Today:

1. *Study guide*
2. *Pens/pencils*
3. *Notebook*

**Today's Study Location:** *The desk in my room*

**Study Time Today:** *From 7:00 p.m. to 8:00 p.m. with a short break at 7:30 p.m.*

(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. “Doing time” at your desk doesn’t count as real studying.)

**If I Start to Get Tired or Lose Focus Today, I Will:** *Do some sit-ups*

**Today's Study Goals and Accomplishments:** (Be specific. Include things like number of pages, units, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small “chunks” or blocks of material at a time.)

<i>Study Task</i>	<i>Completed</i>	<i>Needs More Work</i>	<i>Needs More Information</i>
<i>1. Review what I learned last time</i>	X		
<i>2. Study the first main topic in Unit 1</i>	X		
<i>3. Study the second main topic in Unit 1</i>		X	

#### What I Learned Today:

1. *Reviewed basic functions*
2. *The importance of checking that the answer “makes sense” by estimating first*
3. *How to use math symbols*

**Today's Reward for Meeting My Study Goals:** *Eating some popcorn*

## Appendix D

### Blank Daily Study Plan Sheet

#### Materials I May Need Today:

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_
4. \_\_\_\_\_
5. \_\_\_\_\_

**Today's Study Location:** \_\_\_\_\_

**Study Time Today:** \_\_\_\_\_

(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. “Doing time” at your desk doesn’t count as real studying.)

**If I Start To Get Tired or Lose Focus Today, I Will:** \_\_\_\_\_

**Today's Study Goals and Accomplishments:** (Be specific. Include things like number of pages, sections, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small “chunks” or blocks of material at a time.)

<i>Study Task</i>	<i>Completed</i>	<i>Needs More Work</i>	<i>Needs More Information</i>
1.			
2.			
3.			
4.			
5.			

#### What I Learned Today:

1. \_\_\_\_\_
2. \_\_\_\_\_
3. \_\_\_\_\_

**Today's Reward for Meeting My Study Goals:** \_\_\_\_\_