GPS GEOMETRY

Study Guide

Georgia End-Of-Course Tests

DEPARTMENT OF EDUCATION
STATE OF GEORGIA
CONSTITUTION
WISDOM JUSTICE MODERATION
1776

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Georgia Department of Education
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INTRODUCTION

This study guide is designed to help students prepare to take the Georgia End-of-Course Test (EOCT) for ***GPS Geometry***. This study guide provides information about the EOCT, tips on how to prepare for it, and some suggested strategies students can use to perform their best.

What is the EOCT? The EOCT program was created to improve student achievement through effective instruction and assessment of the standards in the Georgia Performance Standards (GPS). The EOCT program helps to ensure that all Georgia students have access to a rigorous curriculum that meets high performance standards. The purpose of the EOCT is to provide diagnostic data that can be used to enhance the effectiveness of schools' instructional programs.

The Georgia End-of-Course Testing program is a result of the A+ Educational Reform Act of 2000, O.C.G.A. §20-2-281. This act requires that the Georgia Department of Education create end-of-course assessments for students in grades 9 through 12 for the following core high school subjects:

Mathematics

- Mathematics I: Algebra/Geometry/Statistics
- Mathematics II: Geometry/Algebra II/Statistics

--OR--

- GPS Algebra
- GPS Geometry

Social Studies

- United States History
- Economics/Business/Free Enterprise

Science

- Biology
- Physical Science

English Language Arts

- Ninth Grade Literature and Composition
- American Literature and Composition

Getting started: The HOW TO USE THE STUDY GUIDE section on page 6 outlines the contents in each section, lists the materials you should have available as you study for the EOCT, and suggests some steps for preparing for the ***GPS Geometry*** **EOCT**.

HOW TO USE THE STUDY GUIDE

This study guide is designed to help you prepare to take the ***GPS Geometry*** **EOCT**. It will give you valuable information about the EOCT, explain how to prepare to take the EOCT, and provide some opportunities to practice for the EOCT. The study guide is organized into three sections. Each section focuses on a different aspect of the EOCT.

The OVERVIEW OF THE EOCT section on page 8 gives information about the test: dates, time, question format, and number of questions that will be on the ***GPS Geometry*** **EOCT**. This information can help you better understand the testing situation and what you will be asked to do.

The PREPARING FOR THE EOCT section that begins on page 9 provides helpful information on study skills and general test-taking skills and strategies. It explains how to prepare before taking the test and what to do during the test to ensure the best test-taking situation possible.

The TEST CONTENT section that begins on page 15 explains what the ***GPS Geometry*** **EOCT** specifically measures. When you know the test content and how you will be asked to demonstrate your knowledge, it will help you be better prepared for the EOCT. This section also contains some sample EOCT test questions, helpful for gaining an understanding of how a standard may be tested.

With some time, determination, and guided preparation, you will be better prepared to take the ***GPS Geometry*** **EOCT**.

GET IT TOGETHER

In order to make the most of this study guide, you should have the following:

Materials:

* This study guide
* Pen or pencil
* Highlighter
* Paper

Resources:

* Classroom notes
* Mathematics textbook
* A teacher or other adult

Study Space:

* Comfortable (but not too comfortable)
* Good lighting
* Minimal distractions
* Enough work space

Time Commitment:

* When are you going to study?
* How long are you going to study?

Determination:

* Willingness to improve
* Plan for meeting goals

SUGGESTED STEPS FOR USING THIS STUDY GUIDE

1 Familiarize yourself with the structure and purpose of the study guide. (You should have already read the INTRODUCTION and HOW TO USE THE STUDY GUIDE. Take a few minutes to look through the rest of the study guide to become familiar with how it is arranged.)

2 Learn about the test and expectations of performance. (Read OVERVIEW OF THE EOCT.)

3 Improve your study skills and test-taking strategies. (Read PREPARING FOR THE EOCT.)

4 Learn what the test will assess by studying each unit and the strategies for answering questions that assess the standards in the unit. (Read TEST CONTENT.)

5 Answer the sample test question at the end of each lesson. Check your answer against the answer given to see how well you did. (See TEST CONTENT.)

OVERVIEW OF THE EOCT

Good test takers understand the importance of knowing as much about a test as possible. This information can help you determine how to study and prepare for the EOCT and how to pace yourself during the test. The box below gives you a snapshot of the ***GPS Geometry*** **EOCT** and other important information.

THE EOCT AT A GLANCE

Administration Dates:

The EOCT has three primary annual testing dates: once in the spring, once in the summer, and once in the winter. There are also mid-month, online tests given in August, September, October, November, February, and March, as well as retest opportunities within the year.

Administration Time:

Each EOCT is composed of two sections, and students are given 60 minutes to complete each section. There is also a short stretch break between the two sections of the test.

Question Format:

All the questions on the EOCT are multiple-choice.

Number of Questions:

Each section of the ***GPS Geometry*** **EOCT** contains 31 questions; there are a total of 62 questions on the ***GPS Geometry*** **EOCT**.

Impact on Course Grade:

For students in grade 10 or above beginning the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 85% and the EOCT score 15%. For students in grade 9 beginning the 2011–2012 school year and later, the final grade in each course is calculated by weighing the course grade 80% and the EOCT score 20%. A student must have a final grade of at least 70 to pass the course and to earn credit toward graduation.

If you have additional administrative questions regarding the EOCT, please visit the Georgia Department of Education Web site at www.doe.k12.ga.us, see your teacher, or see your school test coordinator.

PREPARING FOR THE EOCT

WARNING!

You cannot prepare for this kind of test in one night. Questions will ask you to apply your knowledge, not list specific facts. Preparing for the EOCT will take time, effort, and practice.

To do your best on the ***GPS Geometry*** **EOCT**, it is important that you take the time necessary to prepare for this test and develop those skills that will help you take the EOCT.

First, you need to make the most of your classroom experiences and test preparation time by using good **study skills**. Second, it is helpful to know general **test-taking strategies** to ensure that you will achieve your best score.

Study Skills

A LOOK AT YOUR STUDY SKILLS

Before you begin preparing for this test, you might want to consider your answers to the following questions. You may write your answers here or on a separate piece of paper.

1. How would you describe yourself as a student?
   Response: ____
2. What are your study skills strengths and/or weaknesses as a student?
   Response: ____
3. How do you typically prepare for a mathematics test?
   Response: ____
4. Are there study methods you find particularly helpful? If so, what are they?
   Response: ____
5. Describe an ideal study situation (environment).
   Response: ____
6. Describe your actual study environment.
   Response: ____
7. What can you change about the way you study to make your study time more productive?
   Response: ____

Effective study skills for preparing for the EOCT can be divided into three categories:

- **Time Management**
- **Organization**
- **Active Participation**

Time Management

Do you have a plan for preparing for the EOCT? Often students have good intentions for studying and preparing for a test, but without a plan, many students fall short of their goals. Here are some strategies to consider when developing your study plan:

- Set realistic goals for what you want to accomplish during each study session and chart your progress.
- Study during your most productive time of the day.
- Study for reasonable amounts of time. Marathon studying is not productive.
- Take frequent breaks. Breaks can help you stay focused. Doing some quick exercises (e.g., sit-ups or jumping jacks) can help you stay alert.
- Be consistent. Establish your routine and stick to it.
- Study the most challenging test content first.
- For each study session, build in time to review what you learned in your last study session.
- Evaluate your accomplishments at the end of each study session.
- Reward yourself for a job well done.

Organization

You don't want to waste your study time. Searching for materials, trying to find a place to study, and debating what and how to study can all keep you from having a productive study session. Get organized and be prepared. Here are a few organizational strategies to consider:

- Establish a study area that has minimal distractions.
- Gather your materials in advance.
- Develop and implement your study plan (see Appendices A–D for sample study plan sheets).

Active Participation

Students who actively study will learn and retain information longer. Active studying also helps you stay more alert and be more productive while learning new information. What is active studying? It can be anything that gets you to interact with the material you are studying. Here are a few suggestions:

- Carefully read the information and then DO something with it. Mark the important points with a highlighter, circle it with a pen, write notes on it, or summarize the information in your own words.
- Ask questions. As you study, questions often come into your mind. Write them down and actively seek the answers.
- Create sample test questions and answer them.
- Find a friend who is also planning to take the test and quiz each other.

Test-taking Strategies

There are many test-taking strategies that you can use before and during a test to help you have the most successful testing situation possible. Below are a few questions to help you take a look at your test-taking skills.

A LOOK AT YOUR TEST-TAKING SKILLS

As you prepare to take the EOCT, you might want to consider your answers to the following questions. You may write your answers here or on your own paper.

1. How would you describe your test-taking skills?
   Response: ____

2. How do you feel when you are taking a test?
   Response: ____

3. List the strategies that you already know and use when you are taking a test.
   Response: ____

4. List test-taking behaviors you use that contribute to your success when preparing for and taking a test.
   Response: ____

5. What would you like to learn about taking tests?
   Response: ____

Suggested Strategies to Prepare for the EOCT

Learn from the past. Think about your daily/weekly grades in your mathematics classes (past and present) to answer the following questions:

- In which specific areas of mathematics were you or are you successful?

  Response: ____

- Is there anything that has kept you from achieving higher scores?

  Response: ____

- What changes should you implement to achieve higher scores?

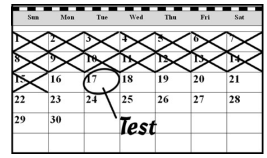
  Response: ____

Before taking the EOCT, work toward removing or minimizing any obstacles that might stand in the way of performing your best. The test preparation ideas and test-taking strategies in this section are designed to help guide you to accomplish this.

Be prepared. The best way to perform well on the EOCT is to be prepared. In order to do this, it is important that you know what standards/skills will be measured on the ***GPS Geometry*** **EOCT** and then practice understanding and using those standards/skills. The TEST CONTENT section of this study guide is designed to help you understand the specific standards that are on the ***GPS Geometry*** **EOCT** and give you suggestions for how to study the standards that will be assessed. Take the time to read through this material and follow the study suggestions. You can also ask your math teacher for any suggestions he or she might offer on preparing for the EOCT.

Start now. Don't wait until the last minute to start preparing. Begin early and pace yourself. By preparing a little bit each day, you will retain the information longer and increase your confidence level. Find out when the EOCT will be administered, so you can allocate your time appropriately.

Suggested Strategies the Day before the EOCT



✓ **Review what you learned from this study guide**.

1. Review the general test-taking strategies discussed in the TOP 10 SUGGESTED STRATEGIES DURING THE EOCT on page 14.
2. Review the content information discussed in the TEST CONTENT section beginning on page 15.
3. Focus your attention on the main topic, or topics, that you are most in need of improving.

✓ **Take care of yourself**.

1. Try to get a good night's sleep. Most people need an average of eight hours, but everyone's sleep needs are different.
2. Don't drastically alter your routine. If you go to bed too early, you might lie in bed thinking about the test. You want to get enough sleep so you can do your best.

Suggested Strategies the Morning of the EOCT

Eat a good breakfast. Choose foods high in protein for breakfast (and for lunch if the test is given in the afternoon). Some examples of foods high in protein are peanut butter, meat, and eggs. Protein gives you long-lasting, consistent energy that will stay with you through the test to help you concentrate better. Avoid foods high in sugar content. It is a misconception that sugar sustains energy—after an initial boost, sugar will quickly make you more tired and drained. Also, don't eat too much. A heavy meal can make you feel tired. So think about what you eat before the test.

Dress appropriately. If you are too hot or too cold during the test, it can affect your performance. It is a good idea to dress in layers, so you can stay comfortable, regardless of the room temperature, and keep your mind on the EOCT.

Arrive for the test on time. Racing late into the testing room can cause you to start the test feeling anxious. You want to be on time and prepared.

TOP 10

Suggested Strategies during the EOCT

These general test-taking strategies can help you do your best during the EOCT.

1 **Focus on the test.** Try to block out whatever is going on around you. Take your time and think about what you are asked to do. Listen carefully to all the directions.

2 **Budget your time.** Be sure that you allocate an appropriate amount of time to work on each question on the test.

3 **Take a quick break if you begin to feel tired.** To do this, put your pencil down, relax in your chair, and take a few deep breaths. Then, sit up straight, pick up your pencil, and begin to concentrate on the test again. Remember that each test section is only 60 minutes.

4 **Use positive self-talk.** If you find yourself saying negative things to yourself such as “I can’t pass this test,” it is important to recognize that you are doing this. Stop and think positive thoughts such as “I prepared for this test, and I am going to do my best.” Letting the negative thoughts take over can affect how you take the test and your test score.

5 **Mark in your test booklet.** Mark key ideas or things you want to come back to in your test booklet. Remember that only the answers marked on your answer sheet will be scored.

6 **Read the entire question and the possible answer choices.** It is important to read the entire question so you know what it is asking. Read each possible answer choice. Do not mark the first one that “looks good.”

7 **Use what you know.** Draw on what you have learned in class, from this study guide, and during your study sessions to help you answer the questions.

8 **Use content domain-specific strategies to answer the questions.** In the TEST CONTENT section, there are a number of specific strategies that you can use to help improve your test performance. Spend time learning these helpful strategies, so you can use them while taking the test.

9 **Think logically.** If you have tried your best to answer a question but you just aren’t sure, use the process of elimination. Look at each possible answer choice. If it doesn’t seem like a logical response, eliminate it. Do this until you’ve narrowed down your choices. If this doesn’t work, take your best educated guess. It is better to mark something down than to leave it blank.

10 **Check your answers.** ☑ When you have finished the test, go back and check your work.

A WORD ON TEST ANXIETY

It is normal to have some stress when preparing for and taking a test. It is what helps motivate us to study and try our best. Some students, however, experience anxiety that goes beyond normal test “jitters.” If you feel you are suffering from test anxiety that is keeping you from performing at your best, please speak to your school counselor, who can direct you to resources to help you address this problem.

TEST CONTENT

Up to this point in this study guide, you have been learning various strategies on how to prepare for and take the EOCT. This section focuses on what will be tested. It also includes sample questions that will let you apply what you have learned in your classes and from this study guide.

This section of the study guide will help you learn and review the various mathematical concepts that will appear on the ***GPS Geometry*** **EOCT**. Since *mathematics* is a broad term that covers many different topics, the state of Georgia has divided it into three major areas of knowledge called **content strands**. The content strands are broad categories. Each of the content strands is broken down into big ideas. These big ideas are called **content standards** or just standards. Each content strand contains standards that cover different ideas related to the content strand. Each question on the EOCT measures an individual standard within a content strand.

The three content strands for the ***GPS Geometry*** **EOCT** are Geometry, Data Analysis and Probability, and Algebra. They are important for several reasons. Together, they cover the major skills and concepts needed to understand and solve mathematical problems. These skills have many practical applications in the real world. Another more immediate reason that the content strands are important has to do with test preparation. The best way to prepare for any test is to study and know the material measured on the test.

This study guide is organized in six **units** that cover the content standards outlined by unit on the GPS Geometry curriculum map. Each unit is presented by topic rather than by specific strand or standard (although those are listed at the beginning of each unit and are integral to each topic). The more you understand about the topics in each unit, the greater your chances of getting a good score on the EOCT.

Studying the Content Standards and Topics (Units 1 through 6)

You should be familiar with many of the content standards and topics that follow. It makes sense to spend more time studying the content standards and topics that you think may cause you problems. Even so, do not skip over any of them. The TEST CONTENT section has been organized into six units. Each unit is organized by the following features:

- **Introduction:** an overview of what will be discussed in the unit
- **Key Standards:** information about the specific standards that will be addressed

  (NOTE: The names of the standards may not be the exact names used by the Georgia Department of Education.)
- **Main Topics:** the broad subjects covered in the unit

  Each Main Topic includes:
  - **Key Ideas:** definitions of important words and ideas as well as descriptions, examples, and steps for solving problems.
  - **Review Examples:** problems with solutions showing possible ways to answer given questions
  - **EOCT Practice Items:** sample multiple-choice questions similar to test items on the ***GPS Geometry*** **EOCT** with answer keys provided

With some time, determination, and guided preparation, you will be better prepared to take the ***GPS Geometry*** **EOCT**.

Unit 1: Geometry Gallery

In this unit, students will explore, understand, and use the language of reasoning and justification, which includes formal discussion of the logical relationships among an implication and its converse, its inverse, and its contrapositive. Students will also use logical reasoning and proofs; prove conjectures through multiple forms of justification; explore angles, triangle inequalities, congruences, and points of concurrency; and apply properties to determine special quadrilaterals.

KEY STANDARDS

MM1G3. Students will discover, prove, and apply properties of triangles, quadrilaterals, and other polygons.

a. Determine the sum of interior and exterior angles in a polygon.
b. Understand and use the triangle inequality, the side-angle inequality, and the exterior-angle inequality.
c. Understand and use congruence postulates and theorems for triangles (SSS, SAS, ASA, AAS, and HL).
d. Understand, use, and prove properties of and relationships among special quadrilaterals: parallelogram, rectangle, rhombus, square, trapezoid, and kite.
e. Find and use points of concurrency in triangles: incenter, orthocenter, circumcenter, and centroid.

MM1G2. Students will understand and use the language of mathematical argument and justification.

a. Use conjecture, inductive reasoning, deductive reasoning, counterexamples, and indirect proof as appropriate.
b. Understand and use the relationships among a statement and its converse, inverse, and contrapositive.

ANGLE PROPERTIES AND RELATIONSHIPS OF ANGLES AND SIDES OF POLYGONS

KEY IDEAS

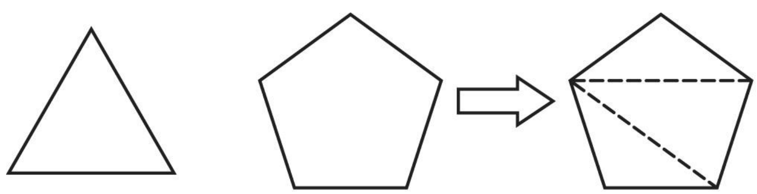
1. In a ***regular polygon***, all side lengths are congruent, and all angles are congruent.

2. The following information applies to ***interior and exterior angles*** of polygons:

   - The ***Interior Sum Theorem*** for triangles states that the sum of the measures of the three interior angles of a triangle always equals 180°.

   - The ***sum of the measures of the interior angles*** of a convex polygon is found by solving $180°(n - 2)$.

     Convex polygons can be divided into triangles.
     In a triangle, $n = 3$, so $(n-2)180° = (3-2)180° = 180°$.
     In a pentagon, $n = 5$, so $(n-2)180° = (5-2)180° = 3(180°) = 540°$.



   - The ***measure of each interior angle*** of a regular n-gon is found by solving $\dfrac{180°(n-2)}{n}$.

   - The ***exterior angle of a polygon*** is an angle that forms a linear pair with one of the angles of the polygon.

   - Interior angles and their adjacent exterior angles are always ***supplementary***. The sum of the degree measures of the two angles is 180°.

   - The ***remote interior angles of a triangle*** are the two angles nonadjacent to the exterior angle.

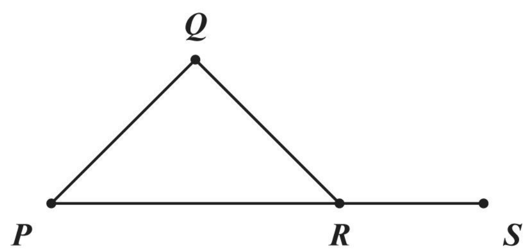
   - The ***measure of the exterior angle of a triangle*** equals the sum of the measures of the two remote interior angles.

- The ***Exterior Angle Inequality*** states that an exterior angle of a triangle is greater than either of the remote interior angles.

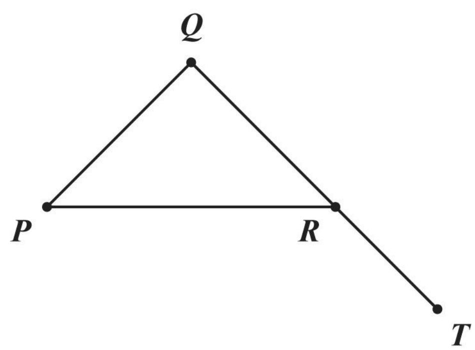
- The ***Exterior Angle Sum Theorem*** states that if a polygon is convex, then the sum of the measures of the exterior angles, one at each vertex, is 360°. The corollary that follows states that the measure of each exterior angle of a regular *n*-gon is $\frac{360°}{n}$.

Example:

If the side of a triangle or a polygon is extended, an angle adjacent to the interior angle is formed. This angle is called an exterior angle of the polygon. In this figure, side $\overline{PR}$ has been extended, $\angle QRP$ is the interior angle, and $\angle QRS$ is the corresponding exterior angle.



In this figure, side $\overline{QR}$ was extended instead of side $\overline{PR}$. In this case, $\angle QRP$ is the interior angle, and $\angle PRT$ is the corresponding exterior angle.



There are always two exterior angles at each vertex of a polygon. When the exterior angle of this triangle is either $\angle QRS$ or $\angle PRT$, the remote interior angles are $\angle PQR$ and $\angle QPR$.

Notice that the sum of the measures of the remote interior angles is the same for either $\angle QRS$ or $\angle PRT$ since they both have the same remote interior angles:

$$m\angle QRS = m\angle PQR + m\angle QPR \text{ and } m\angle PRT = m\angle PQR + m\angle QPR$$

Example:

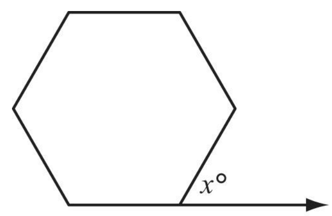
What is the measure, in degrees, of an interior angle of a regular hexagon?

Solution:

$$\frac{180(n-2)}{n}=\frac{180(6-2)}{6}=\frac{180(4)}{6}=\frac{720}{6}=120^\circ$$

Example:

Consider this regular hexagon with an exterior angle shown.



What is the value of x?

Solution:

$$x=\frac{360}{n}=\frac{360}{6}=60^\circ$$

The following theorems apply to triangles.

3. ***Theorem*****:** If one side of a triangle is longer than another side, then the angle opposite the longer side has a greater measure than the angle opposite the shorter side.

4. ***Theorem*****:** If one angle of a triangle has a greater measure than another angle, then the side opposite the greater angle is longer than the side opposite the lesser angle.

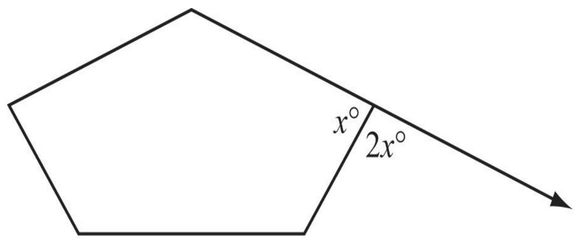
5. ***Theorem*****:** The sum of the lengths of any two sides of a triangle is greater than the length of the third side (the ***Triangle Inequality Theorem***).

6. ***Theorem*****:** If two sides of one triangle are equal to two sides of another triangle but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is larger than the included angle of the second.

7. ***Theorem*****:** If c is the measure of the longest side of a triangle, a and b are the lengths of the other two sides, and $c^2=a^2+b^2$, then the triangle is a right triangle (the ***converse of the Pythagorean theorem***).

REVIEW EXAMPLES

1) What is the degree measure of the exterior angle in this figure?

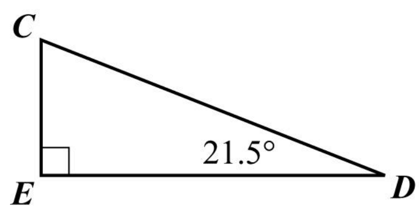


Solution:

In the figure, the number of degrees in the exterior angle of the pentagon is represented by $2x$, and the number of degrees in the adjacent interior angle is represented by x. The exterior angle forms a linear pair with its adjacent interior angle. Therefore,

$x + 2x = 180$
$3x = 180$
$x = 60$, or $60°$, which is the degree measure of the interior angle
The degree measure of the exterior angle is $2x$ or $2(60) = 120°$.

2) Consider $\triangle CDE$.



List the sides in order by length from the greatest to the least.

Solution:

You are given $m\angle E = 90°$ and $m\angle D = 21.5°$. First, use the fact that the sum of the measures of the interior angles of a triangle equals 180º to find $m\angle C$.

$$m\angle C = 180 - 90 - 21.5 = 68.5, \text{ or } 68.5°$$

If an angle of a triangle has a measure greater than another angle, then the side opposite the greater angle is longer than the side opposite the smaller angle.

$$m\angle E > m\angle C > m\angle D\text{; therefore, } \overline{CD} > \overline{DE} > \overline{EC}$$

EOCT Practice Items

1) In $\triangle ECD$, $m\angle E = 136°$, $m\angle C = 17°$, and $m\angle D = 27°$. Which statement must be true?

A. $\overline{CD} < \overline{DE}$
B. $\overline{DE} < \overline{CD}$
C. $\overline{CE} > \overline{CD}$
D. $\overline{DE} > \overline{CE}$

[Key: B]

2) Which set could be the lengths of the sides of a triangle?

A. 15 cm, 18 cm, 26 cm
B. 16 cm, 16 cm, 32 cm
C. 17 cm, 20 cm, 40 cm
D. 18 cm, 22 cm, 42 cm

[Key: A]

3) The first three angles in a pentagon each have the same measure. The other two angles each measure 10° less than each of the first three angles.

What is the measure of one of the first three angles in the pentagon?

A. 102°
B. 104°
C. 112°
D. 114°

[Key: C]

CONGRUENCY

KEY IDEAS

1. The symbol $\cong$ means "is congruent to." If $\triangle ABC \cong \triangle XYZ$, then $\overline{AB} \cong \overline{XY}$, $\overline{BC} \cong \overline{YZ}$, $\overline{AC} \cong \overline{XZ}$, $\angle A \cong \angle X$, $\angle B \cong \angle Y$, and $\angle C \cong \angle Z$.

The following key ideas are all theorems.

2. ***Theorem*****:** If two triangles are congruent, then the corresponding parts of the two congruent triangles are congruent.

3. ***SSS Theorem*****:** If three sides of one triangle are congruent to three sides of another triangle, then the two triangles are congruent.

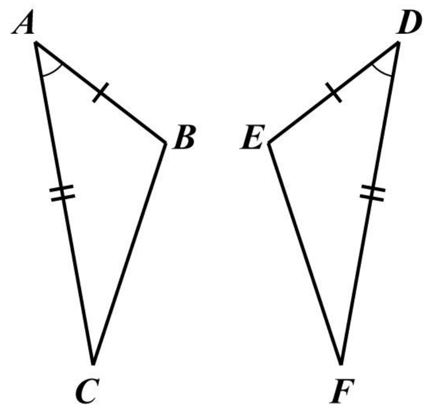
4. ***SAS Theorem*****:** If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

5. ***ASA Theorem*****:** If two angles and the included side of one triangle are equal to two angles and the included side of another triangle, then the two triangles are congruent.
ASA leads to the ***AAS corollary***. If two angles of a triangle are known, then the third angle is also known.

6. ***HL Theorem*****:** If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the two triangles are congruent.

REVIEW EXAMPLES

1) State the theorem that supports that the two triangles are congruent and write the congruence statement. Explain your reasoning.



Solution:

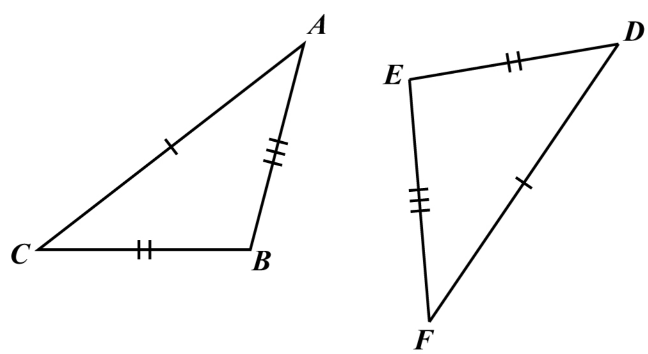
The theorem is SAS. The congruence statement is $\triangle ABC \cong \triangle DEF$.

Reasoning:

- The single tick marks on $\overline{AB}$ and $\overline{DE}$ indicate that the **sides** are **congruent**.
- The two tick marks on $\overline{AC}$ and $\overline{DF}$ indicate that these **sides** are **congruent**.
- The angle markings on $\angle A$ and $\angle D$ indicate that these two **angles** are **congruent**.

The drawing shows two congruent sides and the angles between these two sides are congruent (SAS).

2) What theorem supports that the two triangles are congruent? Complete the statement $\overline{AC} \cong$ ____ .

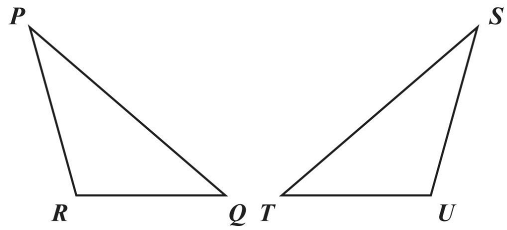


Solution:

By SSS, the two triangles are congruent because the drawing shows three congruent sides. $\triangle ACB \cong \triangle FDE$, so $\overline{AC} \cong \overline{FD}$.

EOCT Practice Items

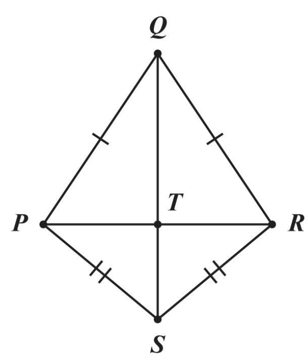
1) Which set of relationships is sufficient to prove that the triangles in this figure are congruent?



A. $\overline{PR} \cong \overline{SU}, \overline{PQ} \cong \overline{ST}, \angle Q \cong \angle U$
B. $\overline{PQ} \cong \overline{PR}, \overline{ST} \cong \overline{SU}, \overline{RQ} \cong \overline{TU}$
C. $\overline{RQ} \cong \overline{TU}, \angle R \cong \angle U, \angle P \cong \angle S$
D. $\angle P \cong \angle S, \angle R \cong \angle U, \angle Q \cong \angle T$

[Key: C]

2) Use this diagram of a kite to answer the question.



Which statement can be proved by using the HL postulate?

A. $\triangle PQR \cong \triangle PSR$
B. $\triangle PTS \cong \triangle TQR$
C. $\triangle QPS \cong \triangle SRQ$
D. $\triangle QTP \cong \triangle QTR$

[Key: D]

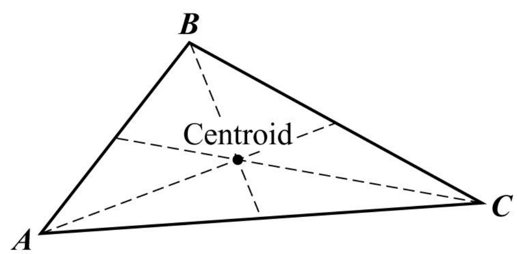
POINTS OF CONCURRENCY IN TRIANGLES

KEY IDEAS

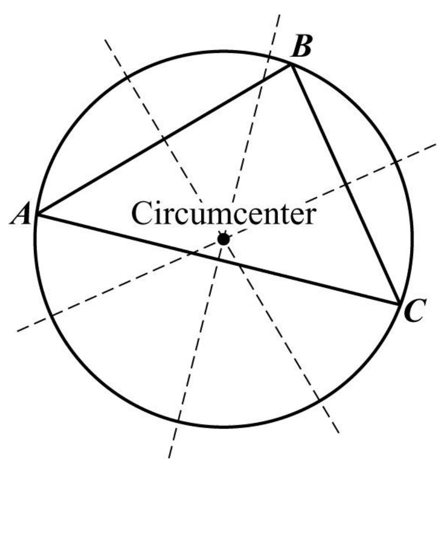
1. Two or more lines that intersect in one point are ***concurrent lines***. This intersection point is known as the ***point of concurrency***.

2. ***Centroid*** is the point of concurrency of the medians of a triangle.

   A ***median*** of a triangle is a segment that joins a vertex of a triangle to the midpoint of the opposite side. The point of concurrency of the medians (the point of intersection) is called the ***centroid*** of the triangle. The centroid of $\triangle ABC$ is shown in this diagram.

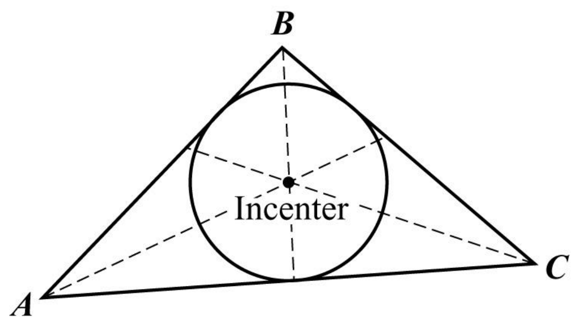


3. ***Circumcenter*** is the point of concurrency of the perpendicular bisectors of the sides of a triangle. This diagram shows that the three perpendicular bisectors of $\triangle ABC$ are concurrent at a single point.



This ***point of concurrency*** is called the ***circumcenter*** of the triangle. This point is also the center of the circle ***circumscribed*** about $\triangle ABC$. Notice that the circle passes through all three vertices of $\triangle ABC$. You can also explain this diagram by stating that $\triangle ABC$ is ***inscribed*** in the circle.

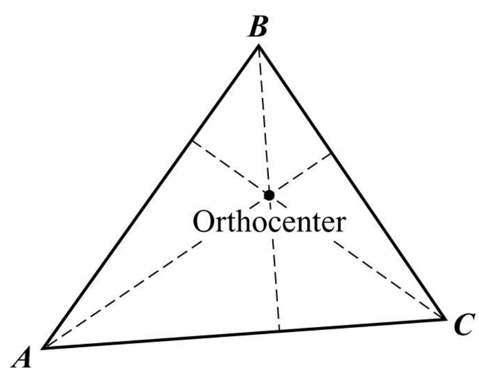
4. ***Incenter*** is the point of concurrency of the bisectors of the angles of a triangle. This diagram shows the angle bisectors of $\triangle ABC$.



The angle bisectors intersect at a point of concurrency known as the ***incenter*** of the triangle. It is the center of the circle that can be inscribed in $\triangle ABC$.

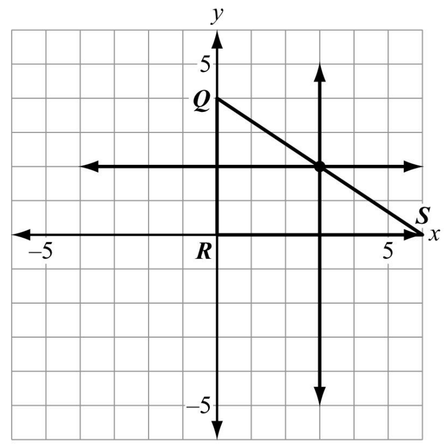
5. ***Orthocenter*** is the point of concurrency of the altitudes of a triangle.

An ***altitude*** of a triangle is a perpendicular segment from a vertex of the triangle to the line containing the opposite side. The ***point of concurrency*** of the lines that contain the altitudes of a triangle (the point of intersection) is called the ***orthocenter*** of the triangle. This diagram shows the orthocenter of $\triangle ABC$.



REVIEW EXAMPLES

1) The vertices of $\triangle QRS$ are located at $Q(0, 4)$, $R(0, 0)$, and $S(6, 0)$.

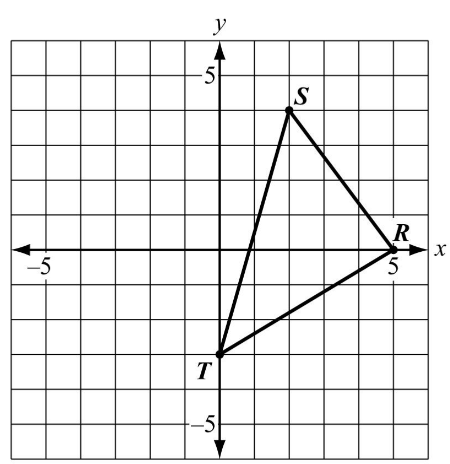


Joe wants to circumscribe a circle about $\triangle QRS$, but he first needs to identify the coordinates of the center of the circle. Use the coordinate grid to identify these coordinates.

Solution:

Two perpendicular bisectors of the sides of $\triangle QRS$ are $x = 3$ and $y = 2$. These lines intersect at (3, 2). Therefore, (3, 2) is the location of the center of the circle to be circumscribed about $\triangle QRS$. (This point is called the circumcenter.)

2) A graphic artist plotted a triangular background for a design on the coordinate grid, as shown.



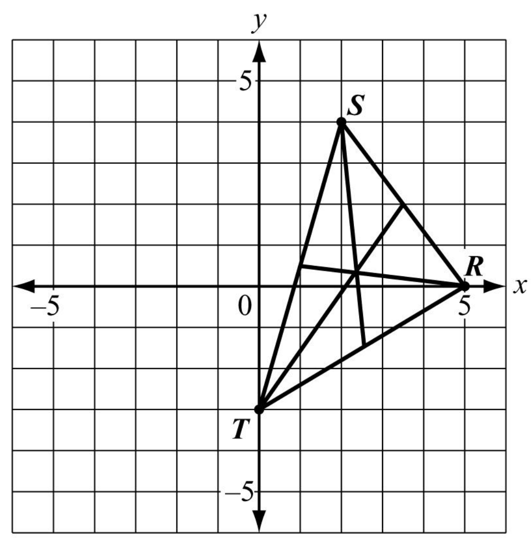
The vertices of $\triangle TRS$ are located at $T(0, -3)$, $R(5, 0)$, and $S(2, 4)$. The artist plans to place an icon as the centroid of the triangle. Identify the coordinates of the centroid of $\triangle TRS$.

Solution:

First, find the midpoint** of each side of $\triangle TRS$.

- Midpoint of $\overline{TR} = \left(\frac{0+5}{2}, \frac{-3+0}{2}\right) = (2.5, -1.5)$
- Midpoint of $\overline{RS} = \left(\frac{5+2}{2}, \frac{0+4}{2}\right) = (3.5, 2)$
- Midpoint of $\overline{ST} = \left(\frac{2+0}{2}, \frac{4+(-3)}{2}\right) = (1, 0.5)$

Use the midpoints to draw the medians of $\triangle TRS$.



The intersection of the medians is the centroid of the triangle. The location of the centroid is $\left(2\frac{1}{3}, \frac{1}{3}\right)$.

****More information regarding the *midpoint formula* can be found in Unit 2 under Key Idea #5 (page 42)**.

EOCT Practice Items

1) A student wants to inscribe a circle inside of a triangle. Which of the following should the student construct to locate the incenter of the triangle?

A. the medians of the triangles
B. the altitudes of the triangles
C. the angle bisectors of the triangle
D. the perpendicular bisectors of the sides of the triangle

[Key: C]

2) Jay constructed a line segment from each vertex that was perpendicular to the line containing the opposite side of a triangle. At what point of concurrency did the lines meet?

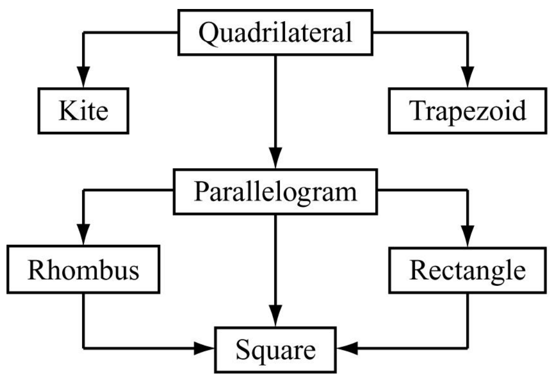
A. the incenter
B. the centroid
C. the orthocenter
D. the circumcenter

[Key: C]

PROPERTIES OF AND RELATIONSHIPS AMONG SPECIAL QUADRILATERALS

KEY IDEAS

1. A polygon with four sides is called a ***quadrilateral***. The special types of quadrilaterals include parallelogram, rectangle, rhombus, square, trapezoid, and kite. The relationship among these figures is shown in this diagram.



It is important to understand the relationship of these figures and their properties in order to properly classify or identify a figure.

Parallelogram
- opposite sides are parallel
- opposite angles are congruent
- opposite sides are congruent
- diagonals bisect each other
- consecutive angles are supplementary

Rhombus
- has all the properties of a parallelogram
- four sides are equal in length
- diagonals are perpendicular
- diagonals bisect each pair of opposite angles

Rectangle
- has all the properties of a parallelogram
- diagonals are congruent
- contains four right angles

Square
- has all the properties of a parallelogram
- diagonals are congruent and perpendicular
- is a rectangle with all sides congruent
- is a rhombus with four right angles

Trapezoid

- one pair of opposite sides that are parallel
- two parallel sides are called bases and the non-parallel sides are the legs
- isosceles trapezoid has one pair of congruent sides and congruent diagonals

Kite

A kite is a quadrilateral that has exactly two distinct pairs of adjacent congruent sides

The following key ideas are all theorems.

2. ***Theorem***: If one pair of opposite sides of a quadrilateral is congruent and parallel, then the quadrilateral is a parallelogram.

3. ***Theorem***: If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

4. ***Theorem***: If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

5. ***Theorem***: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

6. ***Theorem***: If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

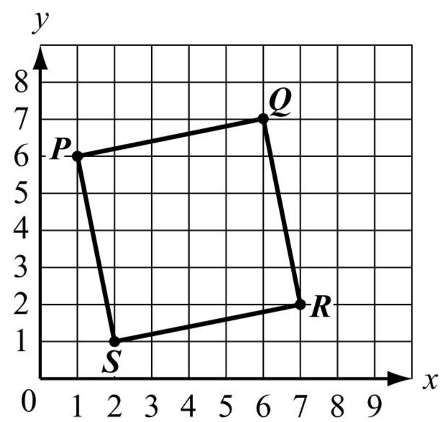
7. ***Theorem***: If each diagonal of a parallelogram bisects a pair of opposite angles, then the parallelogram is a rhombus.

8. ***Theorem***: If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

9. ***Theorem***: If three parallel lines cut off equal segments on one transversal, then they cut off equal segments on every transversal.

REVIEW EXAMPLES

1) The vertices of quadrilateral *PQRS* are plotted at $P(1, 6)$, $Q(6, 7)$, $R(7, 2)$, and $S(2, 1)$.



Prove that *PQRS* is a square.

Solution:

One way to prove it is a square is by proving that the two diagonals are congruent and perpendicular.

Use the distance formula** $\left(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right)$ to show that $\overline{PR} \cong \overline{QS}$.

- The length of $\overline{PR} = \sqrt{(7-1)^2 + (2-6)^2} = \sqrt{36+16} = \sqrt{52}$.
- The length of $\overline{QS} = \sqrt{(6-2)^2 + (7-1)^2} = \sqrt{16+36} = \sqrt{52}$.

Since both lines have the same length, the lines are congruent.

Find the slopes of $\overline{PR}$ and $\overline{QS}$ to show that they are perpendicular.

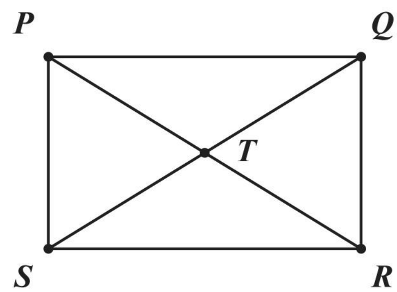
- The slope of $\overline{PR} = \dfrac{6-2}{1-7} = \dfrac{4}{-6} = -\dfrac{2}{3}$.
- The slope of $\overline{QS} = \dfrac{7-1}{6-2} = \dfrac{6}{4} = \dfrac{3}{2}$.

Since the product of the slopes is -1, the lines are perpendicular.

Since both lines are congruent and perpendicular, quadrilateral *PQRS* must be a square.

****More information regarding the *distance formula* can be found in Unit 2 under Key Idea #3 (page 41)**.

2) Rectangle $PQRS$ is shown in this diagram.



The length of $\overline{SR}$ is 30 centimeters. The length of $\overline{ST}$ is 17 centimeters.
What is the perimeter of the rectangle?

Solution:

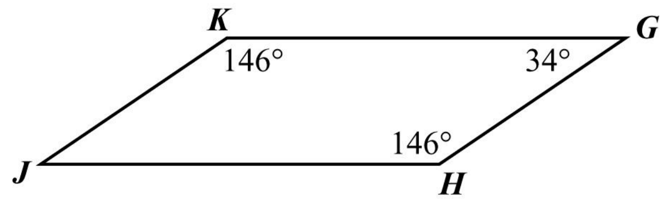
The diagonals of a rectangle bisect each other, which means the length of the diagonal is 34 centimeters. The angles of a rectangle are right angles, so use triangle SQR and the Pythagorean theorem to find the length of side $\overline{QR}$.

$$a^2 + b^2 = c^2$$
$$a^2 + 30^2 = 34^2$$
$$a^2 + 900 = 1156$$
$$a^2 = 256$$
$$a = 16$$

The perimeter is $2l + 2w$ or 2(30) + 2(16) = 60 + 32 = 92, or 92 centimeters.

EOCT Practice Items

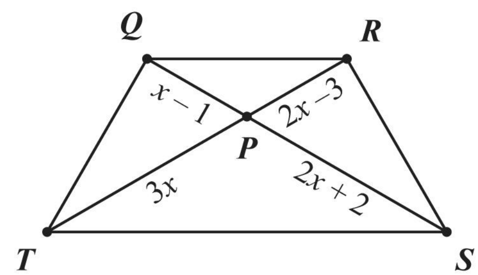
1) Which of the following proves that quadrilateral *GHJK* is a parallelogram?



A. $\angle G$ is supplementary to $\angle H$ and $\angle K$
B. $\angle J$ is complementary to $\angle G$
C. $GH \perp HJ$ and $JK \perp KG$
D. $\overline{GH} \cong \overline{HJ} \cong \overline{JK} \cong \overline{KG}$

[Key: A]

2) This diagram shows isosceles trapezoid *QRST*.



What is the length, in units, of $\overline{QS}$?

A. 2
B. 6
C. 7
D. 9

[Key: C]

MATHEMATICAL ARGUMENT AND JUSTIFICATION

KEY IDEAS

1. ***Inductive reasoning*** is the process of looking for a pattern and using that pattern to make a generalization.

2. ***Deductive reasoning*** is the process of coming to a logical conclusion based on an accepted hypothesis.

3. A ***counterexample*** is an example that shows a conjecture to be false. You need only one counterexample to prove that a conjecture is not true.

4. An ***indirect proof*** is based on eliminating all possible conclusions except for one, resulting in the understanding that the one that is left must be true. It is a proof based on contradiction.

5. A ***statement*** is a sentence that is either true or false, but not both.

6. By definition, a statement is true or false; whether a statement is true or false is called the ***truth value*** of the statement.

7. A ***compound statement***, or ***compound proposition***, is a new statement formed by putting two or more statements together.

8. If p and q are statements, then the statement "if p, then q" is the ***conditional statement***, or implication, with ***hypothesis*** p and ***conclusion*** q. The variables p and q are called statement or ***propositional variables***.

9. If two propositional forms result in statements with the same truth value for all possible cases of substituting statements for the propositional variables, the forms are ***logically equivalent***.

10. Two propositional forms are ***not logically equivalent*** if there exists some group of statements that can be substituted into the propositional forms so that the two statements corresponding to the two forms have different truth values.

11. If p and q are statements, then the statement "p if and only if q" is called a ***biconditional*** statement and is logically equivalent to the statement "if q, then p" and "if p, then q."

12. If a statement is "p," then the ***negation*** of the statement is "not p."

13. In general, a ***conditional statement*** is a statement that can be expressed in “if . . . then” form.

- The ***converse*** of a conditional statement is the new statement obtained by exchanging the hypothesis and the conclusion.

- The ***inverse*** of a conditional statement is the new statement obtained by negating the hypothesis and the conclusion.

- The ***contrapositive*** of a conditional statement is the new statement obtained by both negating and exchanging the hypothesis and the conclusion.

Example:

Statement	If two angles are complementary, then the sum of the measures of the two angles is 90°.
Converse	If the sum of the measures of two angles is 90°, then the two angles are complementary.
Inverse	If two angles are not complementary, then the sum of the measures of the two angles is not 90°.
Contrapositive	If the sum of the measures of two angles is not 90°, then the two angles are not complementary.

14. To say that a conditional statement is true means that whenever the hypothesis is true, then the conclusion is also true; to say that a conditional statement is false means that the hypothesis is, or can be, true while the conclusion is false.

REVIEW EXAMPLES

1) The vertex of $\angle B$ in $\triangle ABC$ forms a vertical angle with $\angle GBH$, and that $m\angle GBH = 64°$ and $m\angle BAC = 73°$.

What conclusions can you draw and support about $\triangle ABC$ from this information?

Solution:

The accepted facts are:

- $\angle B$ in $\triangle ABC$ forms a vertical angle with $\angle GBH$.
- $m\angle GBH = 64°$
- $m\angle BAC = 73°$

Use deductive reasoning to draw conclusions about $\triangle ABC$.

- Since $\angle B$ in $\triangle ABC$ forms a vertical angle with $\angle GBH$, then $m\angle ABC = 64°$, because vertical angles are congruent.

- Since $m\angle BAC = 73°$ and $m\angle ABC = 64°$, then $m\angle ACB = 43°$, because the sum of the angle measures of any triangle is 180º and $m\angle ACB = 180° - 73° - 64°$.
- Since the angle measures of $\triangle ABC$ are all acute angles, 73º, 64º, and 43º, then $\triangle ABC$ is an acute triangle.

2) Samuel wrote this statement.

"The number of feet of the perimeter of a rectangle is always greater than the number of square feet in its area."

What is a counterexample to Samuel's statement?

Solution:

If a rectangle has dimensions of 5 feet by 4 feet, it has a perimeter of 18 feet and an area of 20 square feet. This is a counterexample because it gives an example where Samuel's statement is false. It only takes one counterexample to show that a statement is false.

EOCT Practice Items

1) If David goes to the mall, then his brother will go to the movies. David's brother did not go to the movies.

Assuming that these two statements are true, what conclusion can be drawn?

A. David went to the mall.
B. David went to the movies.
C. David did not go to the mall.
D. David did not go to the movies.

[Key: C]

2) Ella factored the first five out of ten trinomials on a test, and each one factored into a pair of binomials. She made this statement.

"All of the trinomials on this test will factor into a pair of binomials."

Which word or phrase best describes Ella's statement?

A. counterexample
B. inductive reasoning
C. deductive reasoning
D. conditional statement

[Key: B]

Unit 2: Coordinate Geometry

This unit investigates the properties of geometric figures on the coordinate plane. Students develop and use the formulas for the distance between two points, the distance between a point and a line, and the midpoint of segments.

KEY STANDARD

MM1G1. Students will investigate properties of geometric figures in the coordinate plane.

a. Determine the distance between two points.
b. Determine the distance between a point and a line.
c. Determine the midpoint of a segment.
d. Understand the distance formula as an application of the Pythagorean theorem.
e. Use the coordinate plane to investigate properties of and verify conjectures related to triangles and quadrilaterals.

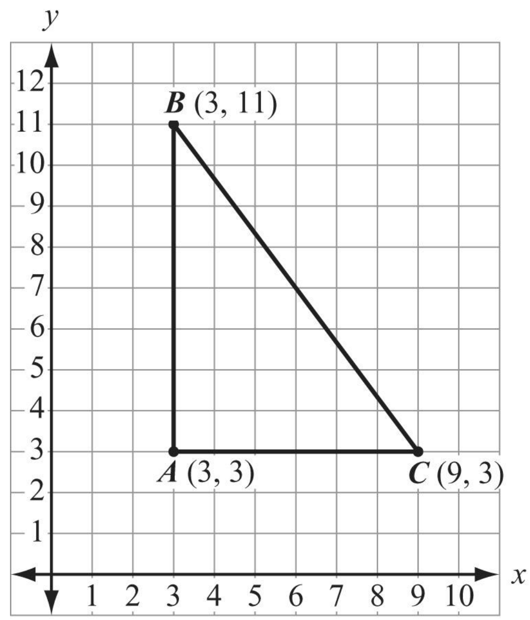
DISTANCES IN THE COORDINATE PLANE

KEY IDEAS

1. On a coordinate plane, the distance between two points that lie on the same vertical line or two points that lie on the same horizontal line can be found by either subtracting the y-coordinates of the two points on the vertical line or the x-coordinates of the two points on the horizontal line and taking the absolute value.

Example:

Consider $\triangle ABC$ shown on this coordinate plane.



To find the distance between points A and B, subtract the y-coordinates of the two points and take the absolute value.

The distance between A and B, or the length of $\overline{AB}$, $= |y_2 - y_1| = |11 - 3| = |8| = 8$, or 8 units.

To find the distance between points A and C, subtract the x-coordinates of the two points and take the absolute value.

The distance between A and C, or the length of $\overline{AC}$, $= |x_2 - x_1| = |3 - 9| = |-6| = 6$, or 6 units.

2. One way to find the distance between two points that are not on the same horizontal or vertical line on a coordinate plane is to use the ***Pythagorean theorem***.

Example:

To find the distance between points B and C in $\triangle ABC$ from Key Idea #1, first notice that $\triangle ABC$ is a right triangle with legs $\overline{AB}$ and $\overline{AC}$ and hypotenuse $\overline{BC}$. The distance between points B and C is also the length of the hypotenuse.

The length of $\overline{AB}$ is 8 units. The length of $\overline{AC}$ is 6 units. Let h represent the length of the hypotenuse. Then use the Pythagorean theorem to calculate its length.

$$h^2 = 8^2 + 6^2$$
$$h^2 = 64 + 36$$
$$h^2 = 100$$
$$h = \sqrt{100}$$
$$h = 10 \text{ units}$$

The distance between B and C, or the length of $\overline{BC}$, is 10 units.

3. Another way to find the distance between two points that are not on the same horizontal or vertical line on a coordinate plane is to use the ***distance formula***, which is an application of the Pythagorean theorem on the coordinate plane. The distance formula states that on a coordinate plane the distance, d, between any two points (x_1, y_1) and (x_2, y_2) is defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example:

To find the distance between B and C, or the length of $\overline{BC}$ from the example in Key Idea #1, use the distance formula and find
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3-9)^2 + (11-3)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$, or 10 units.

The distance formula allows the distance between any two points to be found by using only the coordinates of those two points.

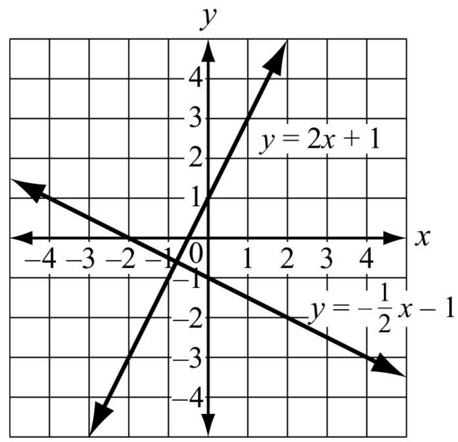
4. A ***perpendicular*** line segment from a point to a line is the shortest segment from the point to the line.

Recall these facts.

- Two lines are perpendicular if they intersect to form four right, or 90º, angles.
- The product of the slopes of two perpendicular lines is –1.

Example:

The lines represented by $y = 2x + 1$ and $y = -\frac{1}{2}x - 1$ are perpendicular.



The two lines intersect to form four right angles.
The product of the slopes of the two lines is $(2)\left(-\frac{1}{2}\right)$, or –1.

5. The midpoint of a line segment is the point that divides the segment into two congruent parts. The ***midpoint formula*** states that the midpoint of a segment with endpoints (x_1, y_1) and (x_2, y_2) is the point that has the coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example:

The coordinates of the endpoints of $\overline{ST}$ are $S(-8, 4)$ and $T(-2, -4)$. To find the coordinates of the midpoint of $\overline{ST}$, substitute the coordinates of the endpoints into the midpoint formula.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\left(\frac{-8 + -2}{2}, \frac{4 + -4}{2}\right)$$

$$\left(\frac{-10}{2}, \frac{0}{2}\right)$$

$$(-5, 0)$$

The midpoint of $\overline{ST}$ is the point with the coordinates (–5, 0).

REVIEW EXAMPLES

1) Quadrilateral *QRST* has vertices at $Q(4,4)$, $R(6,6)$, $S(4,-8)$, and $T(-4,6)$. What is the length of side $\overline{RS}$?

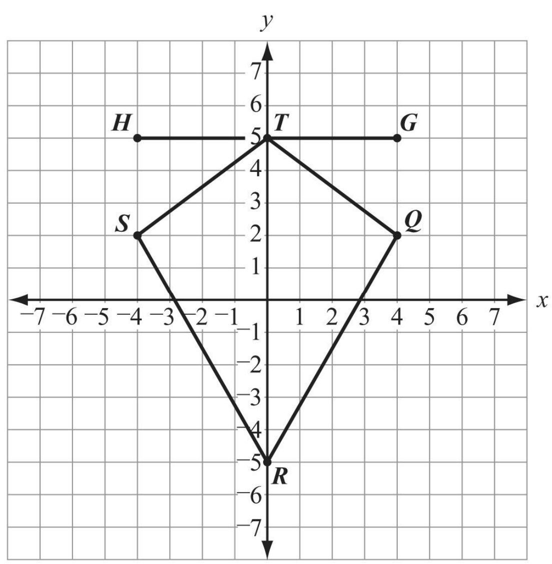
Solution:

Use the two points $R(6,6)$ and $S(4,-8)$ as (x_1, y_1) and (x_2, y_2), respectively. Substitute the coordinates into the distance formula.

$$\overline{RS} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\overline{RS} = \sqrt{(4-6)^2 + (-8-6)^2}$$
$$\overline{RS} = \sqrt{(-2)^2 + (-14)^2}$$
$$\overline{RS} = \sqrt{4+196}$$
$$\overline{RS} = \sqrt{200}$$
$$\overline{RS} = 10\sqrt{2}$$

The length of segment *RS* is $10\sqrt{2}$ units, or approximately 14.1 units.

2) The coordinate grid below shows a sketch of a silver pendant.



Each unit on the grid represents 10 millimeters.

a. What is the shortest distance between point *R* and $\overline{HG}$?

b. How many millimeters longer is $\overline{RT}$ than $\overline{ST}$?

Solution:

a. The shortest distance between point R and $\overline{HG}$ is the length of $\overline{RT}$ because $\overline{RT} \perp \overline{HG}$. Point R is located at (0, –5). Point T is located at (0, 5). To find the distance, subtract the y-coordinates of these two points and take the absolute value of the difference.

The length of $\overline{RT} = |y_2 - y_1| = |5 - (-5)| = |-10| = 10,$ or 10 units.

Since 1 unit = 10 mm, then 10 units = 10 × 10 = 100 mm.
The shortest distance is 100 mm.

b. From part (a), the length of $\overline{RT} = 100$ mm. To find the distance from S to T, use the Pythagorean theorem since the points STH form a right triangle $(\overline{SH} \perp \overline{HT})$.

$$\overline{ST}^2 = \overline{SH}^2 + \overline{TH}^2$$
$$\overline{ST} = \sqrt{\overline{SH}^2 + \overline{TH}^2}$$
$$\overline{ST} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

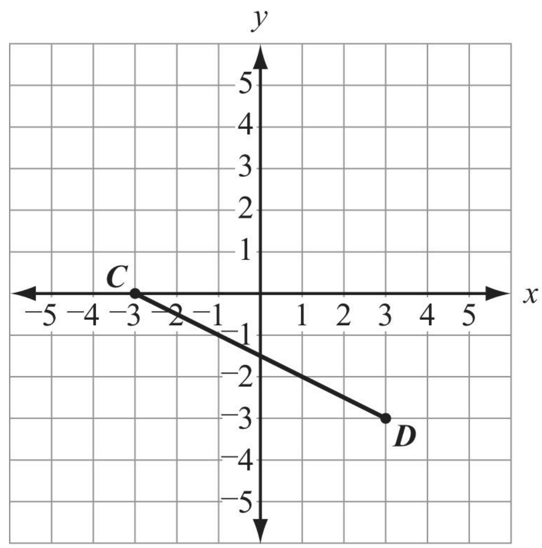
Since 1 unit = 10 mm, then 5 units = 10 × 5 = 50 mm.
So, $\overline{RT} - \overline{ST} = 100 - 50 = 50$ mm.

Note: The length of $\overline{ST}$ can also be found by using the distance formula. The length of $\overline{ST} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Point S is at $(-4, 2)$ and point T is at $(0, 5)$.

The length of $\overline{ST} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(0 - (-4))^2 + (5 - 2)^2} = \sqrt{16 + 9} = 5$ units, or 50 mm.

3) This coordinate grid shows line segment $\overline{CD}$.



Point C is the midpoint of line segment $\overline{BD}$. What are the coordinates of point B?

Solution:

Point C, the midpoint, is located at (−3, 0). Point D, one endpoint of line segment $\overline{BD}$, is located at (3, −3).

Substitute what is known into the x- and y-coordinates of the midpoint formula to find the coordinates of endpoint B.

$$\frac{x_1 + x_2}{2} = -3 \qquad \frac{y_1 + y_2}{2} = 0$$

$$\frac{3 + x_2}{2} = -3 \qquad \frac{-3 + y_2}{2} = 0$$

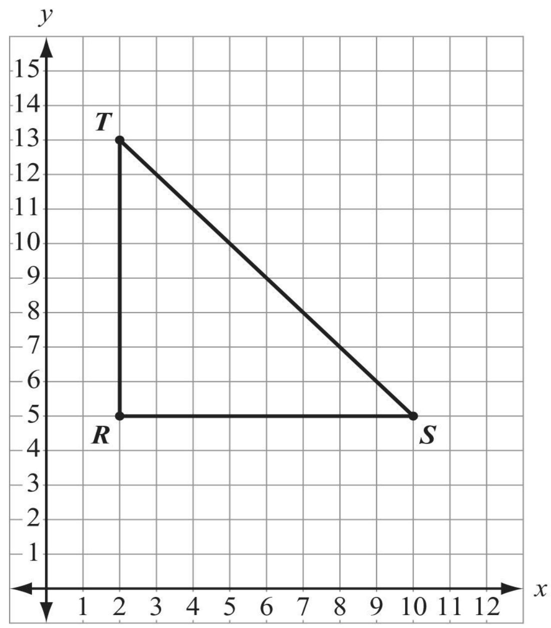
$$3 + x_2 = -6 \qquad -3 + y_2 = 0$$

$$x_2 = -9 \qquad y_2 = 3$$

Endpoint B is located at (–9, 3).

EOCT Practice Items

1) On this coordinate grid, the library is located at point *T*, the music store is located at point *S*, and the pet store is located at point *R*.

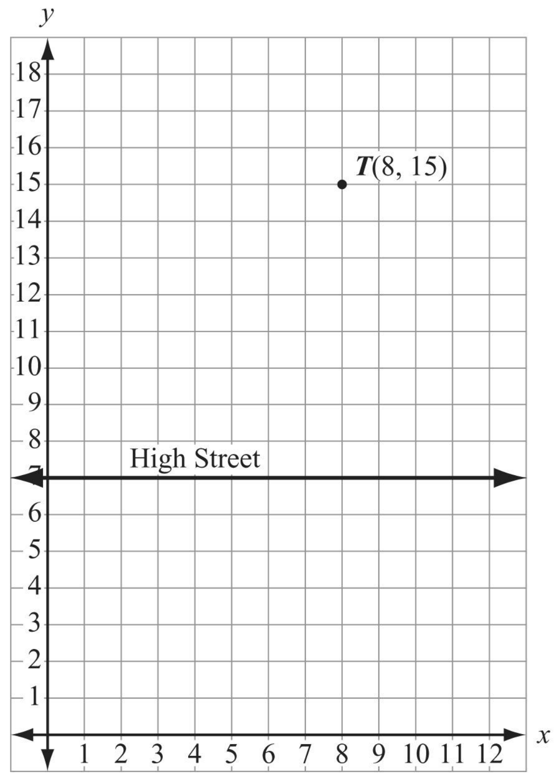


Each grid line represents 1 mile. How much farther, to the nearest tenth of a mile, is the music store from the library than it is from the pet store?

A. 3.3 miles
B. 8.0 miles
C. 11.3 miles
D. 13.6 miles

[Key: A]

2) Wesley is walking with his dog along High Street. He wants to go from High Street to his house located at point *T* on this grid.



Once Wesley leaves High Street, he will have to walk across a rocky field. His dog refuses to walk across the field, so Wesley will have to carry him.

At what point on High Street should Wesley turn and walk toward his house so that he carries his dog the shortest possible distance?

A. (0, 7)
B. (7, 7)
C. (8, 7)
D. (10, 7)

[Key: C]

TRIANGLES AND QUADRILATERALS IN THE COORDINATE PLANE

KEY IDEAS

1. The coordinate plane can be used to determine or prove information regarding various classifications of triangles and quadrilaterals. Coordinate pairs in the coordinate plane identify the locations of points. These locations can be used in the distance and midpoint formulas. Additionally, the points can be used to find the slope. Slope can be used to determine parallel or perpendicular lines.

2. There are many different properties associated with triangles and quadrilaterals. Refer back to Unit 1 of this study guide to review the properties, postulates, and theorems introduced there. The following is a summary of some important properties for your reference.

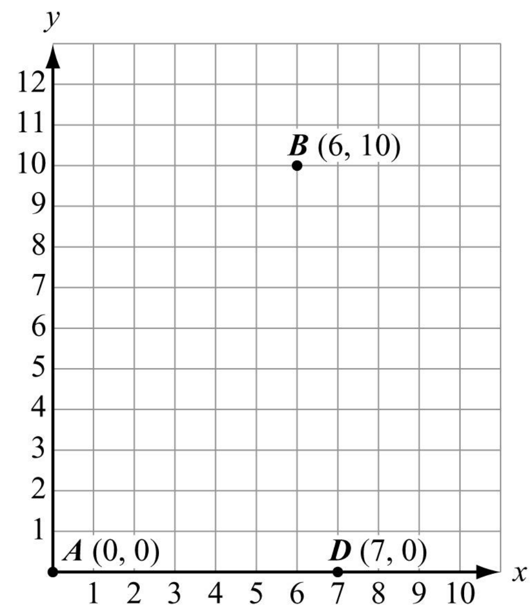
   - Two lines are parallel if their slopes are equal.
   - Two lines are perpendicular if the product of their slopes is –1.
   - Triangles can be classified by their sides (scalene, equilateral, or isosceles) or by their angles (acute, right, or obtuse).
   - Triangles can be proven congruent by using SSS, SAS, ASA, AAS, and HL theorems.
   - The segment joining the midpoints of two sides of a triangle is parallel to the third side, and its length is half the length of the third side.
   - Some quadrilaterals can be classified as squares, rectangles, trapezoids, kites, parallelograms, or rhombuses.
   - If a quadrilateral is a parallelogram, then its opposite sides and opposite angles are congruent, its opposite sides are parallel, its consecutive angles are supplementary, and its diagonals bisect each other.
   - A parallelogram is a rhombus if its diagonals are perpendicular, each of its diagonals bisects a pair of opposite angles, and all four of its sides are congruent.
   - A parallelogram is a rectangle if its diagonals are congruent.
   - A parallelogram is a square if its diagonals are perpendicular and congruent.
   - A quadrilateral is an isosceles trapezoid if its base angles are congruent and its diagonals are congruent.
   - The centroid is the point of concurrency of the medians of a triangle.
   - The circumcenter is the point of concurrency of the perpendicular bisectors of the sides of a triangle.
   - The incenter is the point of concurrency of the bisectors of the angles of a triangle.
   - The orthocenter is the point of concurrency of the altitudes of a triangle.

REVIEW EXAMPLES

1) Three vertices of parallelogram *ABCD* are *A*(0, 0), *B*(6, 10), and *D*(7, 0). What are the coordinates of vertex *C*?

Solution:

Plot what you are given on a coordinate grid.



To determine the coordinates of vertex *C*, think about these facts regarding parallelograms.

- If a quadrilateral is a parallelogram, then its opposite sides and opposite angles are congruent.
- If a quadrilateral is a parallelogram, then its opposite sides are parallel.

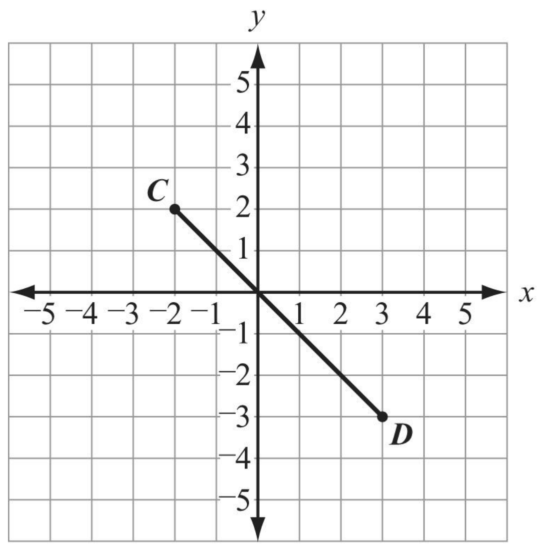
This means that $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$. The length of line segment AD is 7 units. Since $\overline{AD} \cong \overline{BC}$, then the length of line segment $BC = 7$ units. Since $\overline{AD}$ lies on the horizontal axis, then $\overline{BC}$ must also be a horizontal line. The *x*-coordinate of vertex *C* is located 7 horizontal units from the *x*-coordinate of vertex *B*: 6 + 7 = 13.

The *x*-coordinate of vertex *C* is 13.

If *ABCD* is a parallelogram, then $\overline{AD} \parallel \overline{BC}$. Vertex *B* is 10 units from the *x*-axis, so vertex *C* must also be 10 units from the *x*-axis. The *y*-coordinate of vertex *C* is 10.

The coordinates of vertex *C* are (13, 10).

2) Line segment *CD*, shown on this coordinate grid, is the hypotenuse of right isosceles triangle *BCD*.



Identify two possible locations for vertex *B*.

Solution:

Since $\overline{CD}$ is the hypotenuse of $\triangle BCD$, then $\angle B$, the angle opposite the hypotenuse, is a right angle. This means that $\overline{CB} \perp \overline{BD}$. Since the triangle is an isosceles triangle, then $\overline{CB} \cong \overline{BD}$.

Extend a vertical line from point *C* and a horizontal line from point *D*. The lines intersect perpendicularly at point (–2, –3). The length of line segment *CB* is 5 units, and the length of line segment *BD* is 5 units. Since these two lines are the same length, this verifies that the triangle is isosceles.

Extend a horizontal line from point *C* and a vertical line from point *D*. The lines intersect perpendicularly at point (3, 2). The length of line segment *CB* is 5 units, and the length of line segment *BD* is 5 units. This verifies that the triangle is isosceles.

Point *B* could be located at either (–2, –3) or (3, 2).

EOCT Practice Items

1) The vertices of quadrilateral *EFGH* have the coordinates $E(2, -2)$, $F(4, 3)$, $G(-1, 5)$, and $H(-3, 0)$. Which of the following describes quadrilateral *EFGH*?

A. a square
B. a rectangle that is not a square
C. a rhombus that is not a square
D. a parallelogram that is not a rectangle

[Key: A]

2) In isosceles $\triangle PQR$, $\overline{PQ} \cong \overline{QR}$. Point *P* is located at (1, 1). The centroid of the triangle is located at (6, 2).

Which coordinate pair could represent the location of point *R* of the triangle?

A. (1, 4)
B. (5, 1)
C. (7, 3)
D. (11, 1)

[Key: D]

Unit 3: Statistics

This unit investigates analysis and comparison of data sets using mean, standard deviation, and sampling techniques to estimate the mean and standard deviation of populations when the actual mean and standard deviation are not known.

KEY STANDARD

MM2D1. Using sample data, students will make informal inferences about population means and standard deviations.

a. Pose a question and collect sample data from at least two different populations.
b. Understand and calculate the means and standard deviations of sets of data.
c. Use means and standard deviations to compare data sets.
d. Compare the means and standard deviations of random samples with the corresponding population parameters, including those population parameters for normal distributions. Observe that the different sample means vary from one sample to the next. Observe that the distribution of the sample means has less variability than the population distribution.

UNDERSTANDING AND CALCULATING MEANS AND STANDARD DEVIATIONS

KEY IDEAS

1. The ***mean*** of a data set is the mathematical average of the data, which can be found by adding the data values and dividing the sum by the number of data values. The formula for the ***mean*** with *n* data points is $\frac{\sum_{i=1}^{n} X_i}{n}$, where X_i is the i^{th} data point.

   The mean is one of several measures of central tendency that can be used to describe a data set. Its main limitation is that, because every data point directly affects the result, it can be affected greatly by outliers. For example, consider these two sets of quiz scores:

   Student P: {8, 9, 9, 9, 10}
   Student Q: {3, 9, 9, 9, 10}

   Both students consistently performed well on quizzes and both have the same ***median*** and ***mode*** score, 9. Student Q, however, has a mean quiz score of 8, while Student P has a mean quiz score of 9. Although many instructors accept the use of a mean as being fair and representative of a student's overall performance in the context of test or quiz scores, it can be misleading because it fails to describe the ***variation*** in a student's scores and the effect of a single score on the mean can be disproportionately large, especially when the number of scores is small.

2. The ***mean absolute deviation*** of a data set is the mean (or average) of the absolute differences (ignoring the sign) between each data value and the mean value for the set of data. The formula for ***mean absolute deviation*** of a data set is $\frac{\sum_{i=1}^{N}\left|X_i - \bar{X}\right|}{N}$, where X_i is the i^{th} data point and $\bar{X}$ is the mean value of the set of data. The absolute value is used to find the absolute difference, or the magnitude of the difference without respect to whether the difference is positive or negative.

   This is a relatively simple way to describe the variation in a data set that was used in GPS Algebra; it is simply the average value of the distance from the mean for each data point.

   For the two students in Key Idea #1, the mean absolute deviations are as follows:

   Student P: The mean score is 9. The mean absolute deviation is $\frac{1+0+0+0+1}{5} = \frac{2}{5} = 0.4.$ The average distance between the individual data values and the mean value is 0.4. This is a relatively small deviation meaning there is not a lot of variation in the data values.

   Student Q: The mean score is 8. The mean absolute deviation is $\frac{5+1+1+1+2}{5} = \frac{10}{5} = 2.0.$ The average distance between the individual data values and the mean value is 2, which is relatively larger than the mean absolute deviation for student P. There is more variation in the scores for student Q than student P.

3. The ***variance*** of a data set is a measure to quantify the spread of data within a set. The formula for ***variance*** of a data set is $\frac{\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2}{N}$ or $\frac{1}{N}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2$, where X_i is the i^{th} data point, $\bar{X}$ is the mean value of the data set, and N is the number of data points.

4. The ***standard deviation*** of a data set is the square root of the variance, i.e., $\sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2}{N}} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)^2}$, where X_i is the i^{th} data point, $\bar{X}$ is the mean value of the data set, and N is the number of data points.

The standard deviation is more commonly used in statistics than the mean absolute deviation. Originally the preference was due to the use of the absolute value function in the formula for mean absolute deviation, which made calculations of this statistic more complicated than the calculation of the standard deviation. As technology has improved, making the differences in computation of the two statistics less complicated, the habit of using standard deviation as the preferred measure of spread has remained. Variance and standard deviation are the most common ways to describe variation in a data set.

Because the distance of each data point from the mean is squared, slightly more weight is given to data points that are farther from the mean than with the mean absolute deviation. However, in data that has a ***normal distribution***, the majority of data points are relatively close to the mean.

For the two students in Key Ideas #1 and #2, the variance and standard deviation are as follows:

Student P: The variance is $\frac{1+0+0+0+1}{5} = \frac{2}{5} = 0.4.$ The standard deviation is $\sqrt{0.4} = 0.63.$

Student Q: The variance is $\frac{25+1+1+1+4}{5} = \frac{33}{5} = 6.6.$ The standard deviation is $\sqrt{6.6} = 2.57.$

5. A ***normal distribution*** is a set of data that follows a symmetrical, bell-shaped curve. Most of the data is relatively close to the mean. As the distance from the mean increases on both sides of the mean, the number of data points decreases. The ***empirical rule*** states that, for data that is distributed normally,

- approximately 68% of the data will be located within one standard deviation on either side of the mean;
- approximately 95% of the data will be located within two standard deviations on either side of the mean; and
- approximately 99.7% of the data will be located within three standard deviations on either side of the mean.

Important Tip

The extent to which a data set is distributed normally can be gauged by observing what percent of the data falls within one, two, or three standard deviations of the mean.

REVIEW EXAMPLES

1) Jesse is the manager of a guitar shop. He recorded the number of guitars sold each week for a period of 10 weeks. His data is shown in this table.

Week	1	2	3	4	5	6	7	8	9	10
# Guitars Sold	12	15	20	8	15	18	17	21	10	24

a. What is the mean of Jesse's data?

b. What are the variance and the standard deviation of Jesse's data?

c. If an ***outlier*** is defined as any value that is more than two standard deviations from the mean, which, if any, values in Jesse's data would be considered an outlier?

Solution:

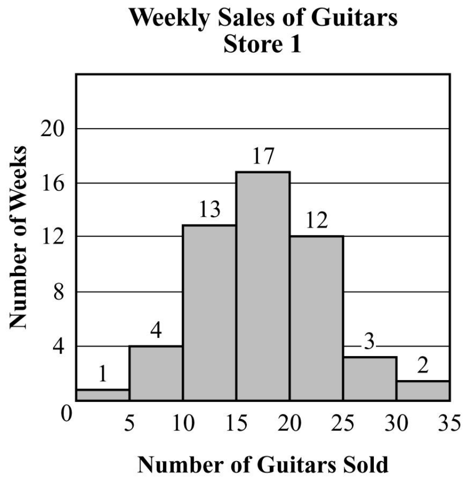
a. The mean is $\frac{(12+15+20+8+15+18+17+21+10+24)}{10}=\frac{160}{10}=16.$

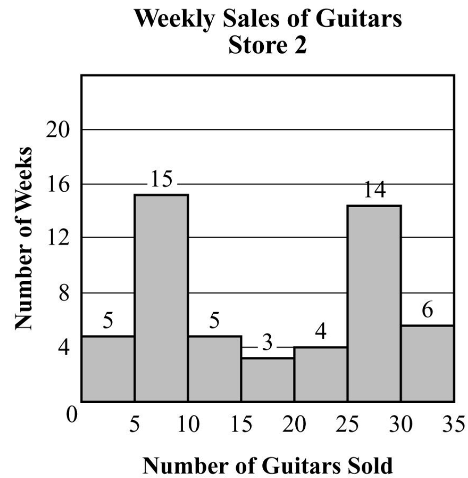
b. The variance is $\frac{(16+1+16+64+1+4+1+25+36+64)}{10}=\frac{228}{10}=22.8.$

The standard deviation is $\sqrt{22.8}=4.77.$

c. $4.77\times 2=9.54,$ so any value less than $16-9.54=6.46$ or greater than $16+9.54=25.54$ is an outlier. There are no such values in Jesse's data.

2) Anne is the regional sales manager for a chain of guitar shops. She recorded the number of guitars sold at two stores in her region each week for one year (52 weeks). These histograms show the data Anne collected.





a. Estimate the mean and standard deviation for each store.

b. What is the range of possible means for each store?

c. Explain why the empirical rule is or is not a good fit for each store.

Solution:

a. Anne's data is presented in a histogram. A histogram shows the ranges into which data points fall, but it does not show individual data points. As a result, both the mean and standard deviation can be estimated but cannot be calculated exactly.

Because the data in each set is symmetrical, the best estimate for the mean is the value in the center of the whole range, 0 to 35, which is 17.5 for each data set.

One way to estimate the variance is to use the middle value in each range and the number of data points in that range. For example, at Store 1 there were 13 weeks in which the number of guitars sold was between 10 and 15. (For now, we won't consider which bar of the graph contains the boundary values 10 and 15.) The middle value of that range is 12.5. The distance of that value from the mean is 5, and the square of that value is 25. So, to calculate the variance, use 13 × 25 for those data points.

Using this technique, the estimated variance for Store 1 is

$$\frac{1\times 225+4\times 100+13\times 25+17\times 0+12\times 25+3\times 100+2\times 225}{52}=\frac{2000}{52}=38.46.$$

The estimated standard deviation is $\sqrt{38.46}=6.20$.

Using the same technique for Store 2, the estimated variance is 107.69, and the estimated standard deviation is 10.38.

b. To consider the range of possible means, we need to know what values are included in each bar. Assume here that each bar includes the boundary value to the right; i.e., the bar between 0 and 5 includes the value 5, and the bar between 5 and 10 includes 10 but not 5. In that case, the minimum possible mean would occur if every value in each bar was the least possible value for that bar. For Store 1, the least possible mean would be

$$\frac{1\times 0+4\times 6+13\times 11+17\times 16+12\times 21+3\times 26+2\times 31}{52}=\frac{831}{52}=15.98.$$

The greatest possible mean would be

$$\frac{1\times 5+4\times 10+13\times 15+17\times 20+12\times 25+3\times 30+2\times 35}{52}=\frac{1040}{52}=20.$$

Using the same strategy, the range of possible means for Store 2 is 15.90 to 20.00.

c. The empirical rule is a good fit for the data for Store 1. The data is symmetrical and most are relatively close to the mean, with a decreasing amount of data as the distance from the mean increases. Using the estimated mean and standard deviation from part a, we would expect 68% of the data to be in the range 17.5 ± 5.4; i.e., 12.1 to 22.9, and 95% of the data to be in the range 17.5 ± 10.8; i.e., 6.7 to 28.3. Although we do not know the exact values for each week, based on the histogram it is reasonable to conclude that 68% of the weeks (35) fall between 12 and 23 guitars sold and 95% of the weeks (49) fall between 7 and 28 guitars sold.

Because the shape of the data distribution for Store 1 approximates a normal distribution, the standard deviation could have been estimated using the empirical rule. If the 35 weeks are within one standard deviation of the estimated mean, 17.5, it would include the 17 weeks in the 15–20 bar and 18 weeks in the 10–15 and 20–25 bars. Those 18 weeks would be split as 9 in each. Again, assuming a normal distribution, we would expect more of the data points in each bar to be closer to the mean, i.e., closer to 17.5, so it is reasonable to estimate that 9 of the data points in the 10–15 bar would have values of 12 or greater and, likewise, that 9 of the data points in the 20–25 bar would have values of 23 or less. This would make 12 one standard deviation less than the mean and 23 one standard deviation greater than the mean, so the estimated standard deviation would be $|12-17.5|$ or $|23-17.5|$, which is 5.5, very close to the standard deviation we estimated in part a.

For Store 2, the empirical rule is not a good fit. Although the data is symmetrical, it is not clustered about the mean as it is for Store 1. This type of data distribution is often called ***bimodal*** as there appears to be two modes, neither of which is close to the value of the mean or the median. Note that the exact median or mode cannot be determined from a histogram, as again the exact values of each data point are unknown.

EOCT Practice Items

1) This table shows the scores of the first six games played in a professional basketball league.

Winning Score	110	98	91	108	109	116
Losing Score	101	88	84	96	77	114

The winning margin for each game is the difference between the winning score and the losing score. What is the standard deviation of the winning margins for these data?

A. 3.8 points
B. 8.3 points
C. 9.5 points
D. 12.0 points

[Key: C]

2) This frequency table shows the heights for Mrs. Quinn's students.

Height (in inches)	Frequency
42	1
43	2
44	4
45	5
46	4
47	2
48	1

What is the approximate standard deviation of these data?

A. 1.0 inches
B. 1.5 inches
C. 2.5 inches
D. 3.5 inches

[Key: B]

SAMPLE MEANS AND DEVIATIONS

KEY IDEAS

1. ***Samples*** are typically used when it is impossible or impractical to collect data for an entire population. To improve the accuracy of the estimate, multiple samples can be drawn from the population. The mean and standard deviation of a set of sample means can be used to estimate the mean and standard deviation of a population when those actual values are not known. For a population with a mean of μ and a standard deviation of σ, the sample means will have a normal distribution with a mean of μ and a standard deviation of $\frac{\sigma}{\sqrt{n}}$, where n represents the number of elements in each sample and σ represents the population standard deviation. This means that, as the sample size increases, the amount of variance among the sample means decreases.

2. Because the population mean is estimated from the mean of a sample and the sample mean is used to calculate the estimated variance and standard deviation for the population, there is some additional variation introduced. For this reason, a better or ***corrected*** estimate of the standard deviation of the population from which a sample is drawn is given by using a denominator of $n-1$ to calculate the standard deviation, i.e., the formula is $S=\sqrt{\frac{\sum_{i=1}^{N}\left(X_i-\overline{X}\right)^2}{N-1}}=\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i-\overline{X}\right)^2}$. This statistic is often called the ***sample standard deviation*** because it is the standard deviation estimated from a sample.

3. The mean and standard deviation of the sample means can be used to estimate the mean and standard deviation of the population. If the mean of the sample means is $\overline{X}$ and the standard deviation of the sample means is $S_{\overline{X}}$, the population mean can be estimated by using $\overline{X}$ and the standard deviation can be estimated by using $S_{\overline{X}}\sqrt{n}$.

REVIEW EXAMPLE

Anne, the regional manager for the chain of guitar shops in the previous review example, took 10 random samples from her data about the number of guitars sold at each of the two shops per week during the last year. The sample means for each shop are as follows:

Shop 1: {21.25, 15.25, 25.0, 15.0, 14.0, 18.0, 12.25, 19.25, 22.0, 21.25}
Shop 2: {17.5, 18.25, 8.0, 22.25, 7.75, 18.25, 24.0, 28.5, 16.0, 16.25}

a. What was the sample size that Anne used?

b. What is the mean and corrected standard deviation for each set of samples?

c. Use the sample mean and corrected standard deviation to estimate the mean and standard deviation for the entire year's data.

Solution:

a. The sample means all come out to numbers that end in (have decimal parts of) .0, .5, .25, or .75. The sample mean is found by dividing the sum of the sample data points, which is an integer, by *n*, the sample size. A divisor of 4 would give the decimal parts above, so the sample size is 4. (Note: It is unlikely that a sample size of a multiple of 4 could also yield these sets of sample means as long as the samples are chosen at random and the data contains values that will form sums containing all the possible remainders when divided by *n*.)

b. Using the formulas given in the previous section,

- for the samples for Store 1, the mean is 18.325 and the corrected standard deviation is 4.11;
- for the samples for Store 2, the mean is 17.675 and the corrected standard deviation is 6.46.

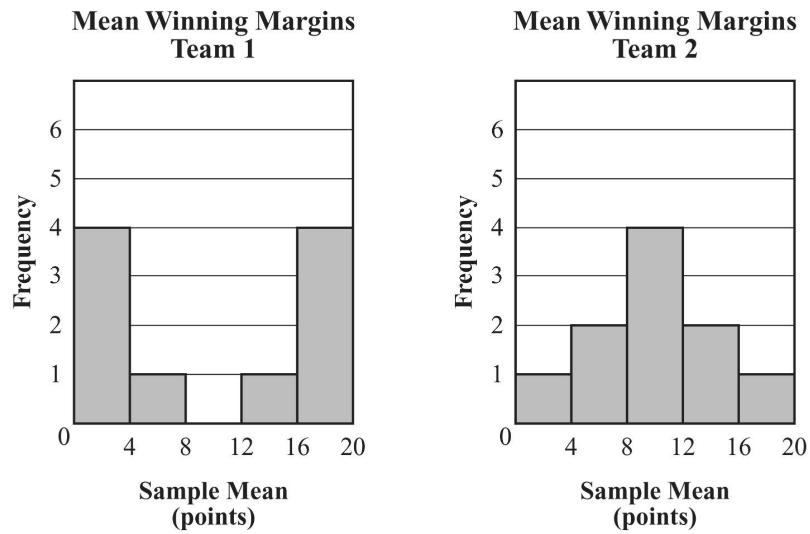
c. Based on the information above, an estimate for the standard deviation of the weekly number of guitars sold for the entire year is $\sigma \times \sqrt{n}$. Because $n = 4$, $\sqrt{n} = 2$. Thus, for Store 1, the estimated standard deviation is 8.22, and for Store 2, the estimated standard deviation is 12.92.

Important Tip

While graphing calculator use is not allowed nor necessary on the EOCT, instructors are likely to use them for classroom experiences. When using a calculator or computer program to compute variance and standard deviation, it is important to know which formula is being used. On calculators such as the TI-84, the ***corrected standard deviation***, or ***sample standard deviation***, is given as S_x and the ***uncorrected standard deviation***, or ***population standard deviation***, is given as σ_x.

EOCT Practice Items

1) Kara took 10 random samples of the winning margins for each of two professional basketball teams. The sample size was 4. The distributions of the sample means are shown in these histograms.

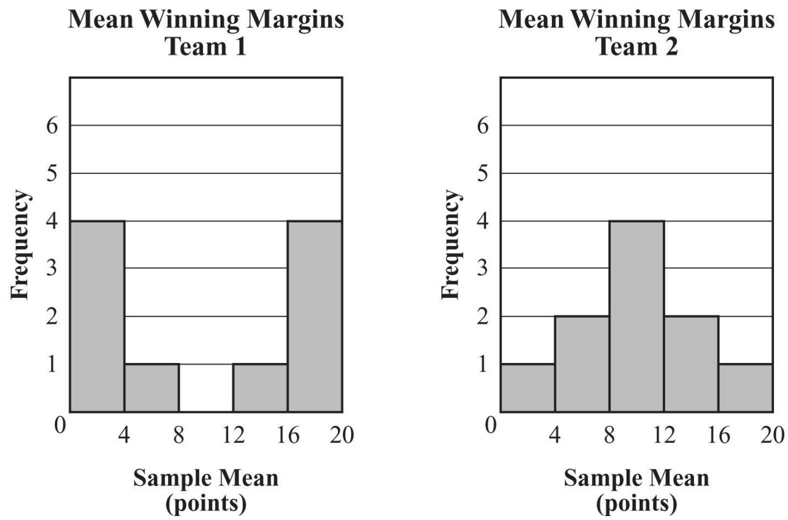


Which is the best estimate of the standard deviation for both samples?

A. Team 1: 3.75 points; Team 2: 2.2 points
B. Team 1: 7.4 points; Team 2: 4.4 points
C. Team 1: 15 points; Team 2: 8.8 points
D. Team 1: 10 points; Team 2: 10 points

[Key: B]

2) John took 10 random samples of the winning margins for each of two professional basketball teams. The distributions of the sample means are shown in these histograms.

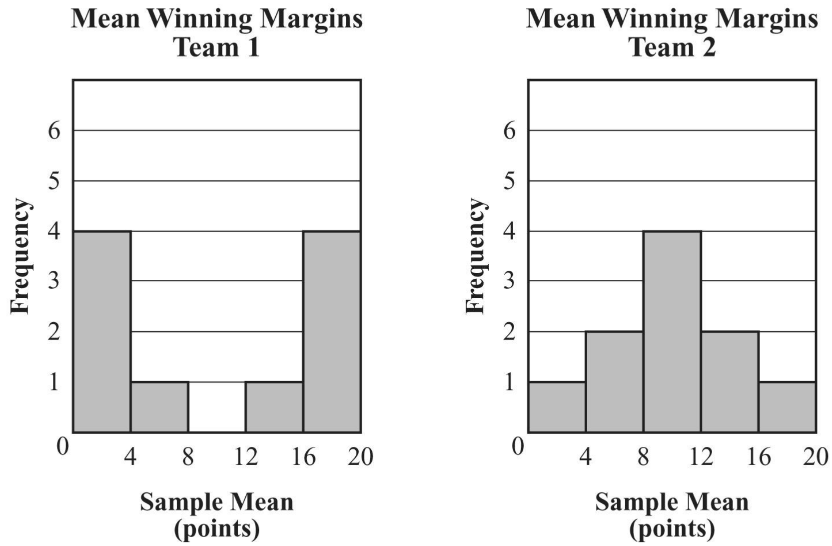


Based on John's data, which statement is MOST likely true?

A. Both the sample mean and the sample standard deviation are greater for Team 1 than for Team 2.
B. The sample means for both teams are equal, but the sample standard deviation for Team 1 is greater.
C. The sample means for both teams are equal, but the sample standard deviation for Team 2 is greater.
D. Both the sample mean and the sample standard deviation are greater for Team 2 than for Team 1.

[Key: B]

3) Mary took 10 random samples of the winning margins for each of two professional basketball teams. The samples were taken from all 82 games in one season. The distributions of the sample means are shown in these histograms.



Which question can be answered based on Mary's data?

A. Which team had the greater number of all-star players?
B. Which team won more games?
C. Which team won by a more consistent margin?
D. Which team lost more games by a narrow margin?

[Key: C]

4) In a set of 10 random samples of winning scores for games played in a professional basketball league, the sample size is 6, the sample mean is 97.5 points, and the sample standard deviation is 5.2 points. Which expression represents the estimated standard deviation of all the winning scores?

A. $\frac{5.2}{\sqrt{10}}$

B. $5.2\sqrt{10}$

C. $\frac{5.2}{\sqrt{6}}$

D. $5.2\sqrt{6}$

[Key: D]

Unit 4: Right Triangle Trigonometry

This unit investigates the properties of right triangles. Relationships between side lengths and angle measures are explored, including properties of 30-60-90 and 45-45-90 triangles and the trigonometric ratios sine, cosine, and tangent.

KEY STANDARDS

MM2G1. Students will identify and use special right triangles.

a. Determine the lengths of sides of 30°-60°-90° triangles.
b. Determine the lengths of sides of 45°-45°-90° triangles.

MM2G2. Students will define and apply sine, cosine, and tangent ratios to right triangles.

a. Discover the relationship of the trigonometric ratios for similar triangles.
b. Explain the relationship between the trigonometric ratios of complementary angles.
c. Solve application problems using the trigonometric ratios.

RIGHT TRIANGLE RELATIONSHIPS

KEY IDEAS

1. Right triangle relationships are all based on the fact that when one of the triangle congruence theorems applies, the relative measures of the angles and sides are fixed. For example, if the length of the hypotenuse and measure of one of the acute angles is known, any triangle with a hypotenuse of that length and an acute angle of that measure will be congruent by the HA theorem. As a result, the length of the other sides and the measures of the other angles can be determined.

2. All right triangles whose acute angles have measures of 30º and 60º are similar due to angle-angle-angle (AAA) similarity. In the case of a right triangle that has acute angles with measures of 30° and 60°, where *s* represents the length of the shorter of the two legs, the measures of the sides are as follows:

   - The length of the shorter leg is *s*.
   - The length of the longer leg is $s\sqrt{3}$.
   - The length of the hypotenuse is $2s$.

3. All right triangles whose acute angles have measures of 45º are similar as well. In the case of a right triangle with acute angles that both measure 45°, where s represents the length of each leg (note that, since the triangle is isosceles, the two legs are congruent), the measures of the sides are as follows:

- The length of each leg is s.
- The length of the hypotenuse is $s\sqrt{2}$.

4. The trigonometric ratios ***sine***, ***cosine***, and ***tangent*** are defined as ratios of the lengths of the sides in a right triangle with a given acute angle measure. These terms are usually seen abbreviated as ***sin***, ***cos***, and ***tan***.

- The ***sine*** of angle A is equal to the ratio $\dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
- The ***cosine*** of angle A is equal to the ratio $\dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
- The ***tangent*** of angle A is equal to the ratio $\dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.
- The tangent of angle A is also equivalent to $\dfrac{\sin A}{\cos A}$.

Important Tip

It is very important to commit these definitions to memory. Many people use the mnemonic "Soh Cah Toa" as a memory prompt.

5. The two acute angles of any right triangle are complementary. Relative to those two angles, the sides that are opposite and adjacent to a given angle are exchanged for the other angle. As a result, if angles P and Q are complementary, $\sin P = \cos Q$ and $\sin Q = \cos P$.

REVIEW EXAMPLES

1) The area of a square is 10 square centimeters. What is the product of the lengths of the diagonals of the square?

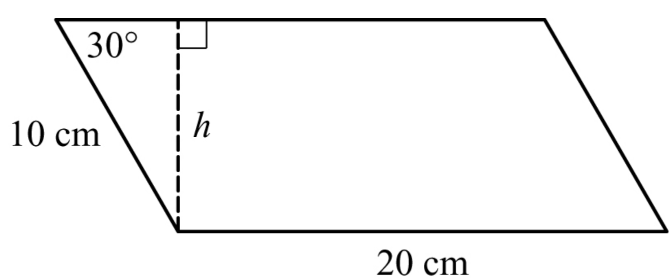
Solution:

Since the square has an area of 10 cm^2, the side lengths can be found by solving $x^2 = 10$, where x represents the length of a side in centimeters. Since only the positive square root has meaning in this context, $x = \sqrt{10}$. The diagonals form angles of 45° with the sides so the length of each diagonal is $\sqrt{10} \bullet \sqrt{2} = \sqrt{20} = 2\sqrt{5}$ cm. The product of the lengths of the diagonals is therefore $2\sqrt{5} \bullet 2\sqrt{5} = 20\ \text{cm}^2$; i.e., the area of the square is equivalent to the half of the product of the lengths of the diagonals.

2) A parallelogram has sides that are 10 cm and 20 cm long. The measure of the acute angles of the parallelogram is 30°. What is the area of the parallelogram?

Solution:

The given information is illustrated in this graphic. To find the area of the parallelogram, we need the length of one of the sides, which was given, and the corresponding altitude, which is shown with a dashed line and labeled *h*.

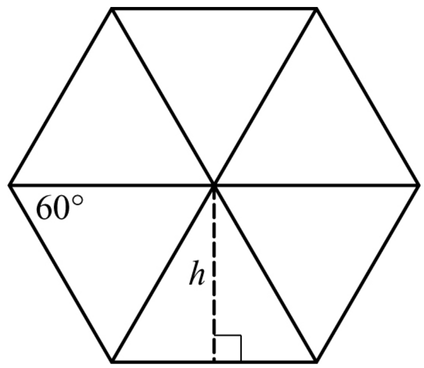


The altitude creates a right triangle in which the side that is 10 cm long is the hypotenuse. The altitude is the shorter of the two legs; therefore, its length is half the length of the hypotenuse, 5 cm. Thus, the area of the parallelogram is $20 \times 5 = 100\ \text{cm}^2$.

3) What is the area of a regular hexagon with sides that are 10 cm long?

Solution:

The given information is illustrated in this graphic. The point where all six small triangles share a vertex is the center of the hexagon. Note that the small triangles are all equilateral so the measure of the angles of the triangles is 60°. To find the area of a triangle, we will need to find the altitude. One altitude of a triangle is marked with a dashed line; the length of the altitude in centimeters is represented by *h*.

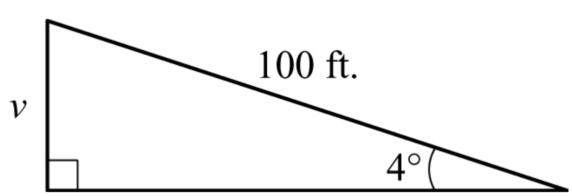


The altitude creates a 30-60-90 triangle in which the length of the hypotenuse is 10 cm and the length of the shorter leg is half of 10, or 5 cm. The altitude, *h*, is $\sqrt{3}$ times the length of the shorter leg, so $h = 5\sqrt{3}$. The area of each of the six equilateral triangles is $\frac{1}{2}(10)\left(5\sqrt{3}\right) = 25\sqrt{3}$; so the area of the hexagon is 6 times that amount, $150\sqrt{3}\ \text{cm}^2$.

4) A road ascends a hill at an angle of 4°. For every 100 feet of road, how many feet does the road ascend?

Solution:

The given information is illustrated in this graphic. The vertical ascent is labeled *v*.



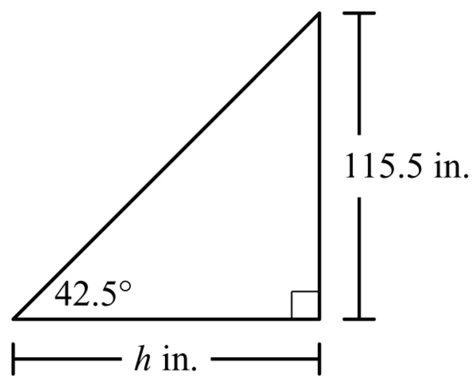
We need the ratio of the opposite side and the hypotenuse, so we will use the sine:

$\sin(4°) = \frac{v}{100} \rightarrow v = 100 \cdot \sin(4°) = 6.976$ feet.

5) According to building codes, the maximum angle of ascent for a staircase in a home is 42.5°. To get from the first floor to the second floor in a new home, a staircase will have a total vertical distance of 115.5 inches. What is the minimum horizontal distance, to the nearest inch, needed for the staircase?

Solution:

The given information is illustrated in this graphic. The horizontal distance is labeled *h*.



We need the ratio of the opposite side and the adjacent side, so we will use the tangent:

$$\tan(42.5°) = \frac{115.5}{h} \rightarrow h = \frac{115.5}{\tan(42.5°)} = 126 \text{ inches} = 10 \text{ feet, } 6 \text{ inches.}$$

EOCT Practice Items

1) The length of one diagonal of a rhombus is 12 cm. The measure of the angle opposite that diagonal is 60º.

What is the perimeter of the rhombus?

A. 24 cm

B. 48 cm

C. $12\sqrt{3}$ cm

D. $24\sqrt{3}$ cm

[Key: B]

2) Angle *J* and angle *K* are complementary angles in a right triangle. The value of tan *J* is $\frac{15}{8}$.

What is the value of sin *J*?

A. $\frac{8}{17}$

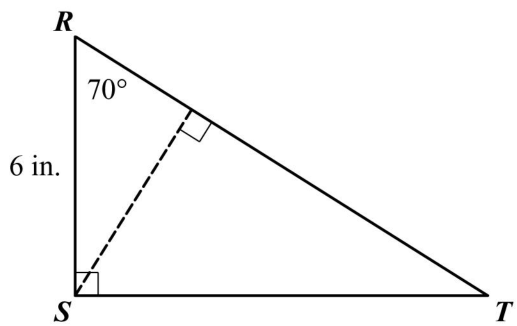
B. $\frac{8}{15}$

C. $\frac{15}{17}$

D. $\frac{17}{15}$

[Key: C]

3) Triangle *RST* is a right triangle with right angle *S*, as shown.



What is the area of triangle *RST*?

A. 6.15 sq. in.
B. 6.54 sq. in.
C. 46.47 sq. in.
D. 49.45 sq. in.

[Key: D]

Unit 5: Circles and Spheres

This unit investigates the properties of circles and spheres. Properties of circles are used to solve problems involving arcs, angles, sectors, chords, tangents, and secants. The surface area and volume of spheres are also addressed.

KEY STANDARDS

MM2G3. Students will understand the properties of circles.

a. Understand and use properties of chords, tangents, and secants as an application of triangle similarity.
b. Understand and use properties of central, inscribed, and related angles.
c. Use the properties of circles to solve problems involving the length of an arc and the area of a sector.
d. Justify measurements and relationships in circles using geometric and algebraic properties.

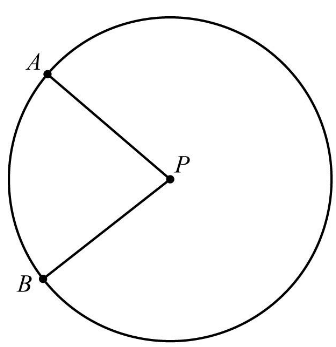
MM2G4. Students will find and compare the measures of spheres.

a. Use and apply surface area and volume of a sphere.
b. Determine the effect on surface area and volume of changing the radius or diameter of a sphere.

PROPERTIES OF CIRCLES: CENTRAL ANGLES AND ARC MEASURES

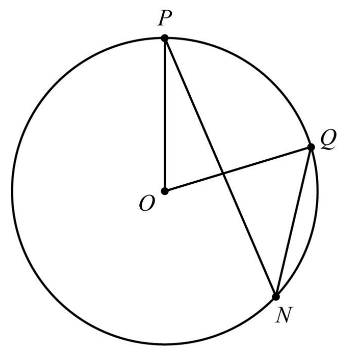
KEY IDEAS

1. The measure of a minor arc of a circle is equal to the measure of the corresponding central angle.



$$m\angle APB = m\overset{\frown}{AB}$$

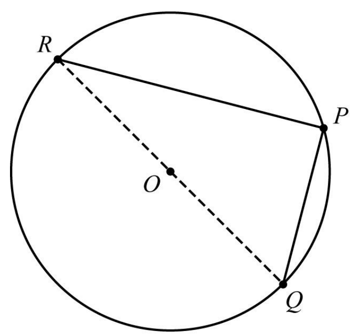
2. The measure of an angle inscribed in a circle, that is, an angle that has its vertex on the circle, is half the measure of the corresponding minor arc, as shown below.



$$m\angle PNQ = \frac{1}{2} m\angle \overset{\frown}{PQ}$$

$$m\angle POQ = m\overset{\frown}{PQ} = 2(m\angle PNQ)$$

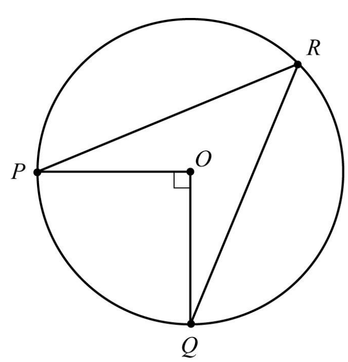
3. An angle inscribed in a semicircle has a measure of 90º, as shown below.



$$m\angle RPQ = \frac{1}{2}\left(m\overset{\frown}{RQ}\right) = \frac{1}{2}(180) = 90°$$

REVIEW EXAMPLES

1) In this circle, what is the measure of $\angle PRQ$?



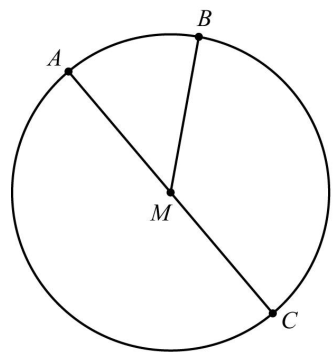
Solution:

The central angle, $\angle POQ$, is a right angle so $m\overset{\frown}{PQ} = 90°$.

$m\angle PRQ = \frac{1}{2} m\overset{\frown}{PQ} = 45°$.

Note that, regardless of where point R is located on the major arc $\overset{\frown}{PQ}$, $m\angle PRQ = 45°$.

2) In Circle M below, $m\angle AMB = 50°$ and $\overline{AC}$ is a diameter. Find $m\overset{\frown}{AB}$, $m\overset{\frown}{ACB}$, and $m\overset{\frown}{BC}$.



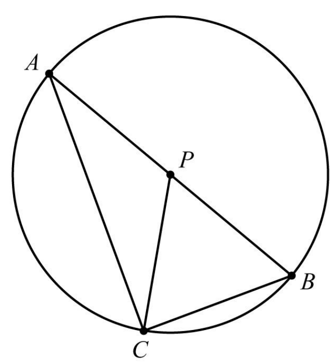
Solution:

Since $\angle AMB$ is a central angle and $m\angle AOB = 50°$, $m\overset{\frown}{AB} = 50°$

$m\overset{\frown}{ACB} = 360 - m\overset{\frown}{AB} = 360 - 50 = 310°$.

Since $\overline{AC}$ is a diameter, $m\overset{\frown}{AC} = 180°$, so $m\overset{\frown}{BC} = 180 - 50 = 130°$.

3) In Circle P below, $\overline{AB}$ is a diameter.



If $m\angle APC = 120°$, find:

a. $m\angle BPC$

b. $m\angle BAC$

c. $m\overset{\frown}{BC}$

d. $m\overset{\frown}{AC}$

Solution:

a. $\angle APC$ and $\angle CPB$ are supplementary, so $m\angle BPC = 180 - 120 = 60°$.

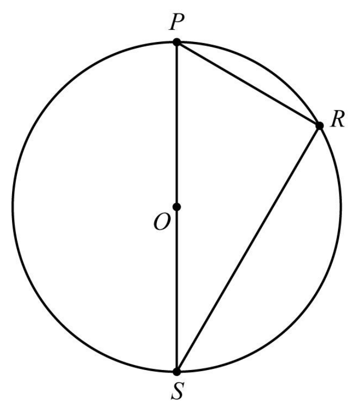
b. Triangle APC is an isosceles triangle, as two legs are formed by radii of the circle. Therefore the two base angles at points A and C are congruent.
$m\angle BAC = \frac{1}{2}(180 - 120) = 30°$.

c. $m\overset{\frown}{BC} = m\angle CPB = 60°$ or $m\overset{\frown}{BC} = 2m\angle BAC = 2 \times 30 = 60°$.

d. $m\overset{\frown}{AC} = m\angle APC = 120°$.

EOCT Practice Items

1) In circle *O*, $\overline{PS}$ is a diameter. The measure of $\overset{\frown}{PR}$ is 72º.



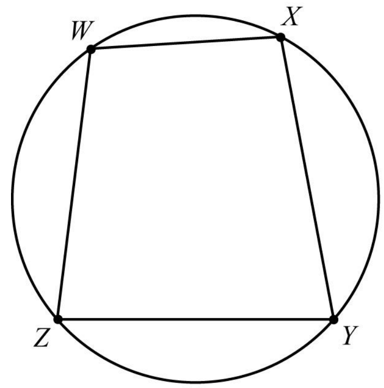
not drawn to scale

What is the measure of $\angle SPR$?

A. 36º
B. 54º
C. 72º
D. 108º

[Key: B]

2) Quadrilateral *WXYZ* is inscribed in this circle.

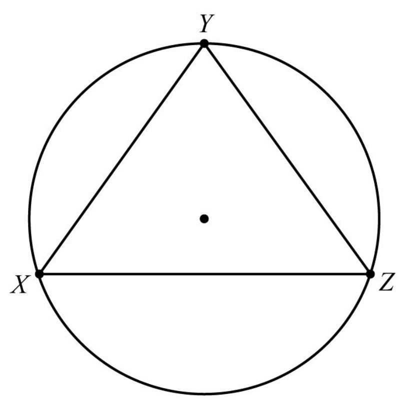


Which statement must be true?

A. $\angle W$ and $\angle Y$ are complementary.
B. $\angle W$ and $\angle Y$ are supplementary.
C. $\angle Z$ and $\angle Y$ are complementary.
D. $\angle Z$ and $\angle Y$ are supplementary.

[Key: B]

3) Isosceles triangle *XYZ* is inscribed in this circle.



- $\overline{XY} \cong \overline{ZY}$
- $m\overset{\frown}{YZ} = 108°$

What is the measure of $\angle XYZ$?

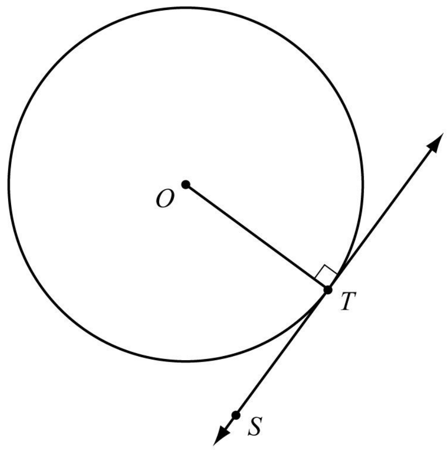
A. 48°
B. 54°
C. 72°
D. 108°

[Key: C]

PROPERTIES OF CIRCLES: TANGENTS, SECANTS, AND CHORDS

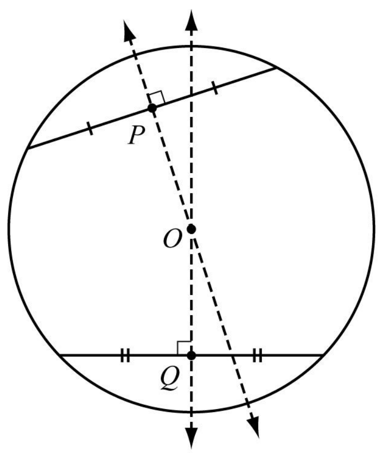
KEY IDEAS

1. A tangent line is perpendicular to the radius of a circle that meets it at the point of tangency, as shown below.



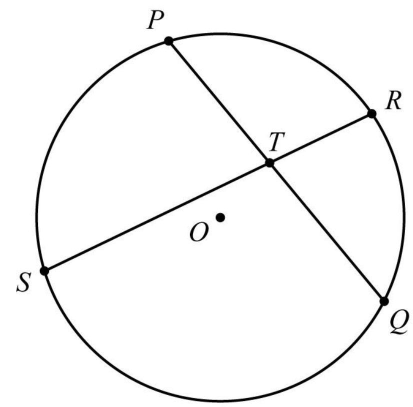
$\overleftrightarrow{ST}$ is tangent to circle O at point T. $\overleftrightarrow{ST} \perp \overline{OT}$.

2. The perpendicular bisector of a chord passes through the center of the circle, as shown below.



The perpendicular bisectors of each chord, $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$, pass through O, the center of the circle.

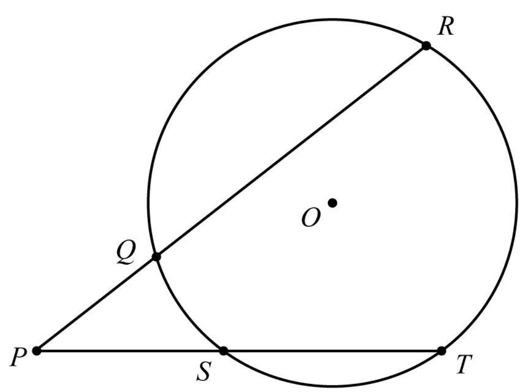
3. If two chords intersect, the angles created have a measure that is equal to half of the sum of the corresponding arc measures, as shown in this circle.



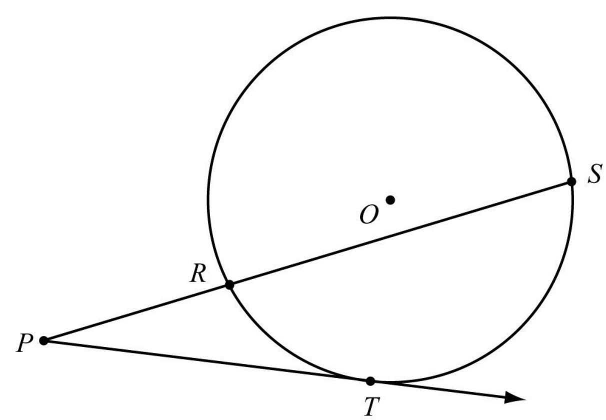
$$m\angle PTR = m\angle STQ = \frac{1}{2}\left(m\overset{\frown}{PR} + m\overset{\frown}{SQ}\right)$$

$$m\angle PTS = m\angle RTQ = \frac{1}{2}\left(m\overset{\frown}{PS} + m\overset{\frown}{RQ}\right)$$

4. The angle created by two secants or by a secant and a tangent has a measure that is equal to half the difference of the corresponding arc measures, as shown in these circles.

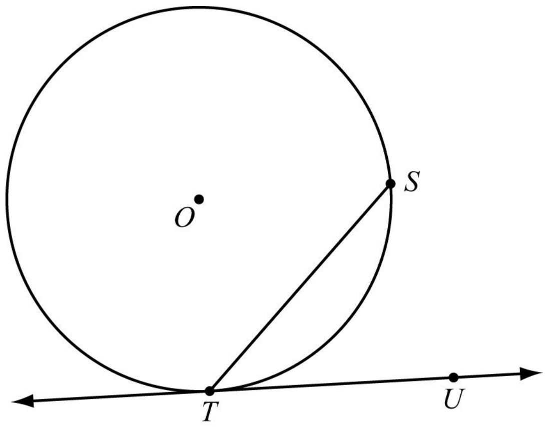


$$m\angle RPT = \frac{1}{2}\left(m\overset{\frown}{RT} - m\overset{\frown}{QS}\right)$$



$$m\angle SPT = \frac{1}{2}\left(m\overset{\frown}{ST} - m\overset{\frown}{RT}\right)$$

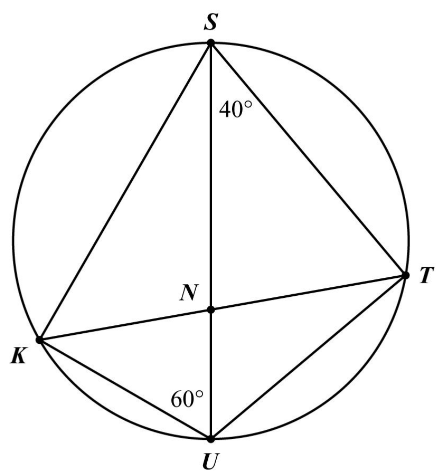
5. The angle created by a tangent and a chord has a measure that is equal to half the measure of its intercepted arc, as shown in this circle.



$$m\angle STU = \frac{1}{2} m\overset{\frown}{ST}$$

REVIEW EXAMPLES

1) Quadrilateral *UTSK* is inscribed in a circle.



Find $m\overset{\frown}{KS}$ and $m\angle KNS$

Solution:

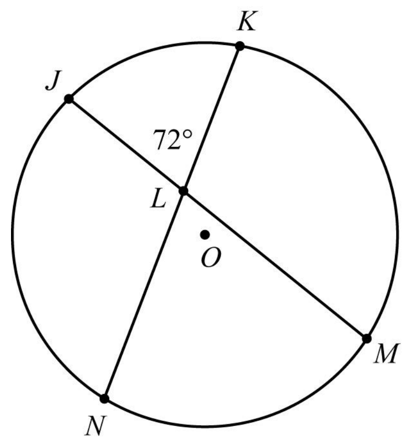
Minor arc $\overset{\frown}{KS}$ is formed by chords $\overline{US}$ and $\overline{UK}$. These chords also form $\angle KUS$.

$m\overset{\frown}{KS} = 2m\angle KUS = 2 \times 60 = 120°$

$m\overset{\frown}{TU} = 2m\angle TUS = 2 \times 40 = 80°$

$m\angle KNS = \frac{1}{2}\left(m\overset{\frown}{KS} + m\overset{\frown}{TU}\right) = \frac{1}{2}(120 + 80) = 100°$

2) In the circle below, chords $\overline{JM}$ and $\overline{KN}$ are congruent. They intersect to form an angle with a measure of 72º. In addition, $m\widehat{MN} = 3m\widehat{JK}$.



Find the measures of minor arcs $\widehat{JK}$, $\widehat{MN}$, $\widehat{JN}$, and $\widehat{KM}$.

Solution:

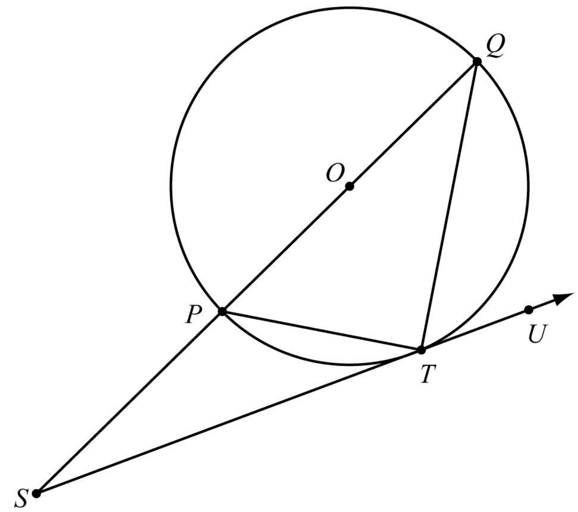
$m\angle JLK = \frac{1}{2}\left(m\widehat{MN} + m\widehat{JK}\right)$; using the given information and letting x represent $m\widehat{JK}$, we have $72 = \frac{1}{2}(3x + x) = \frac{1}{2}(4x) = 2x$, so $x = 36$.

Thus, $m\widehat{JK} = 36°$ and $m\widehat{NM} = 3(36) = 108°$.

Since $\overline{JM} \cong \overline{KN}$, $\widehat{JM} \cong \widehat{KN}$. Both arcs contain $\widehat{JK}$, so by subtracting the common arc from each $\overline{JN} \cong \overline{KM}$. If a set of arcs form a complete circle, the sum of their measures is 360º. Hence, $m\widehat{JK} + m\widehat{KM} + m\widehat{MN} + m\widehat{NJ} = 360$, so $36 + m\widehat{KM} + 108 + m\widehat{NJ} = 360$. But $m\widehat{JN} = m\widehat{KM}$, so this equation is equivalent to $144 + 2\left(m\widehat{KM}\right) = 360$.

$2\left(m\widehat{KM}\right) = 216$, so $m\widehat{KM} = m\widehat{JN} = 108°$.

3) In circle O, $\overline{PQ}$ is a diameter. Point S is on $\overline{PQ}$. $\overrightarrow{SU}$ is tangent to circle O at point T. The measure of $\angle QTU$ is 60º.



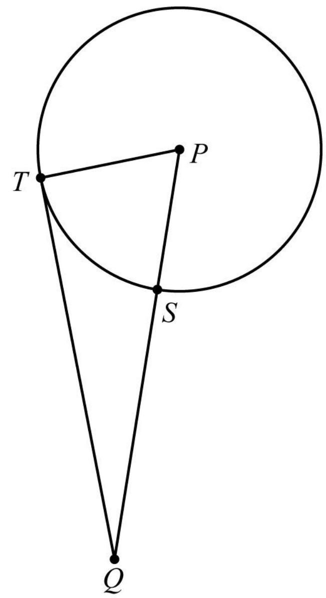
a. Find $m\angle QPT$.
b. Find $m\angle PQT$.
c. Find $m\overset{\frown}{QT}$.
d. Find $m\overset{\frown}{PT}$.
e. Find $m\angle QSU$.
f. Is it possible to construct this figure such that $m\angle QTU = 45°$? Explain why or why not.

Solution:

a. $\angle QPT \cong \angle QTU$ so $m\angle QPT = 60°$.
b. $\angle QTP$ is a right angle so $m\angle PQT = 90 - 60 = 30°$.
c. $m\overset{\frown}{QT} = 2(m\angle QPT) = 120°$.
d. $m\overset{\frown}{PT} = 2(m\angle PQT) = 60°$.
e. $m\angle QSU = \frac{1}{2}\left(m\overset{\frown}{QT} - m\overset{\frown}{PT}\right) = \frac{1}{2}(120 - 60) = 30°$.
f. If $m\angle QTU = 45°$, then $m\angle QPT = 45°$. Since $\angle QPT$ and $\angle TQP$ are complementary, then $m\angle TQP = 45°$ as well. This means that $\angle TQP \cong \angle QTU$, making alternate interior angles congruent, $\overleftrightarrow{PQ} \parallel \overleftrightarrow{TU}$. But if $\overleftrightarrow{PQ} \parallel \overleftrightarrow{TU}$, point S cannot be on both of those lines. Therefore, it is not possible that $m\angle QTU = 45°$.

EOCT Practice Items

1) In this diagram, segment $\overline{QT}$ is tangent to circle P at point T.

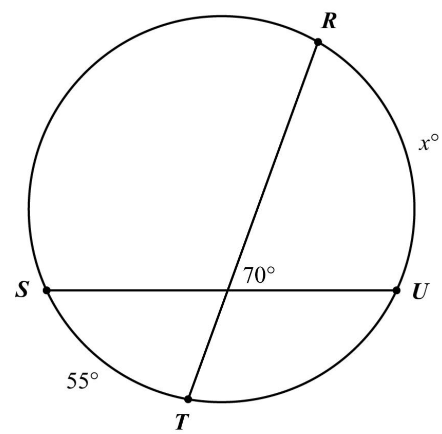


The measure of minor arc $\overset{\frown}{ST}$ is 70º. What is $m\angle TQP$?

A. 20º
B. 25º
C. 35º
D. 40º

[Key: A]

2) Points *R, S, T,* and *U* lie on the circle. The measure of $\overset{\frown}{RU}$ is represented by *x*.

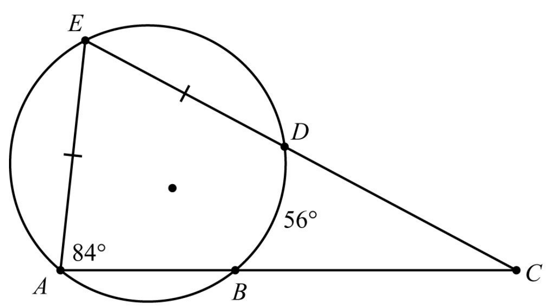


What is the value of *x*?

A. 70
B. 85
C. 110
D. 140

[Key: B]

3) Points *A, B, D* and *E* lie on the circle. Point *C* is outside the circle.



- $\overline{AE} \cong \overline{DE}$
- $m\overset{\frown}{BD} = 56°$
- $m\angle EAC = 84°$

What is the measure of $\angle ACE$?

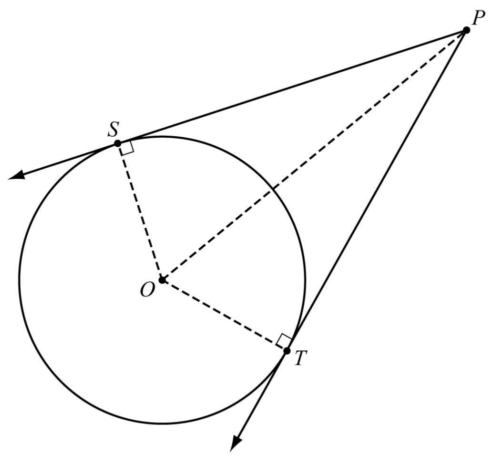
A. 28°
B. 42°
C. 56°
D. 84°

[Key: A]

PROPERTIES OF CIRCLES: LINE SEGMENT LENGTHS

KEY IDEAS

1. From a point outside the circle, the segments from that point to the points of tangency are congruent, as shown below.



$\overline{SP} \cong \overline{TP}$. Note that $\triangle OSP \cong \triangle OTP$. As a result, $\overline{PT} \cong \overline{PS}$ and $\overrightarrow{PO}$ bisects $\angle SPT$.

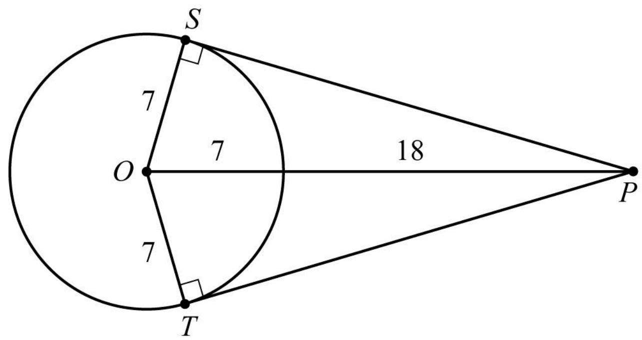
2. Properties of other geometric figures (such as similar triangles, right triangles, or quadrilaterals) can be combined with properties of circles to find lengths of line segments.

REVIEW EXAMPLES

1) A circle has a radius of 7 cm. Point P is located 18 cm outside the circle, with $\overrightarrow{PT}$ and $\overrightarrow{PS}$ tangent to the circle at points T and S, respectively. What are the lengths of $\overline{PT}$ and $\overline{PS}$?

Solution:

This figure illustrates the given information.



Since $\overline{PT} \perp \overline{OT}$ and $\overline{PS} \perp \overline{OS}$, triangles PSO and PTO are right triangles. The Pythagorean theorem can be used to find the lengths PT and PS:

$PS = \sqrt{PO^2 - OS^2} = \sqrt{25^2 - 7^2} = \sqrt{625 - 49} = \sqrt{576} = 24.$

Since $PS = PT$, both PT and PS are 24 cm.

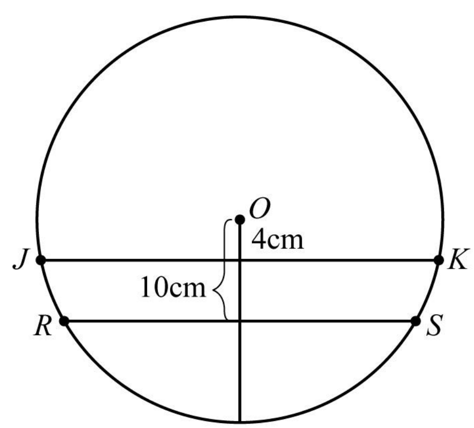
2) The radius of circle O is 20 cm. Chord $\overline{JK}$ is located 4 cm from the center of the circle. Chord $\overline{RS}$ is located 10 cm from the center of the circle.

a. What are the lengths of chords $\overline{JK}$ and $\overline{RS}$?

b. What is the measure of $\angle ORS$?

Solution:

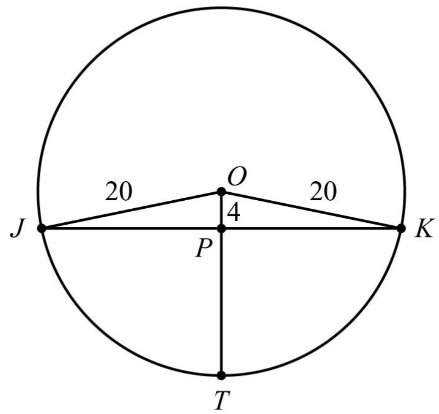
The circle below is a drawing of the situation described.



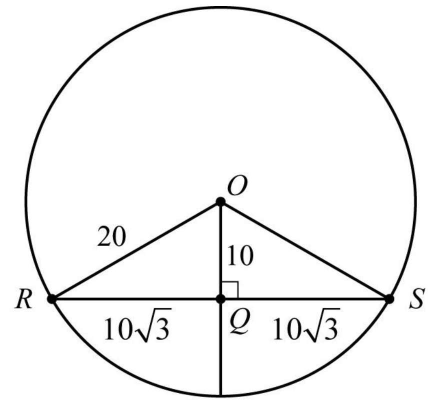
By forming triangles using the chords and the center of the circle, properties of right triangles can be used to find the lengths of the line segments.

Figures A and B illustrate the given information for each chord separately.

A.



B.



a. Chords $\overline{JK}$ and $\overline{RS}$ are perpendicular to radius $\overline{OT}$, so the Pythagorean theorem can be used to find the lengths of JP and RQ:
$JP = \sqrt{20^2 - 4^2} = \sqrt{384} = 8\sqrt{6}$
$RQ = \sqrt{20^2 - 10^2} = \sqrt{300} = 10\sqrt{3}$
Radius $\overline{OT}$ bisects the chords as well as being perpendicular to them, so
$JK = 2\left(8\sqrt{6}\right) = 16\sqrt{6}$ and $RS = 2\left(10\sqrt{3}\right) = 20\sqrt{3}$.

b. The lengths of the sides of triangle ORQ have the relationship $s, s\sqrt{3}, 2s$. This means they form a 30-60-90 triangle. Angle ORS is opposite the shorter leg; therefore, $m\angle ORS = 30°$.

3) Consider Circle *O* with a radius of *r* cm.

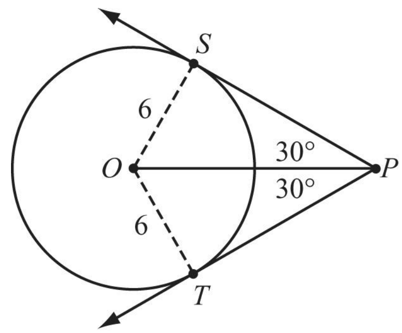
a. If $r = 6$ cm, how far outside the circle is point *P* if the two tangents from *P* to circle *O* form an angle with a measure of 60º?

b. In terms of *r*, how far outside the circle is point *P* if the two tangents from *P* to circle *O* form an angle with a measure of 60º?

c. How far outside the circle is point *P* if the two tangents from *P* to circle *O* form an angle with a measure of 90º?

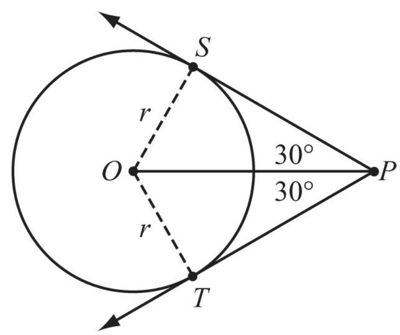
Solution:

a. The figure below illustrates circle *O* and point *P* with the angles described in part a.



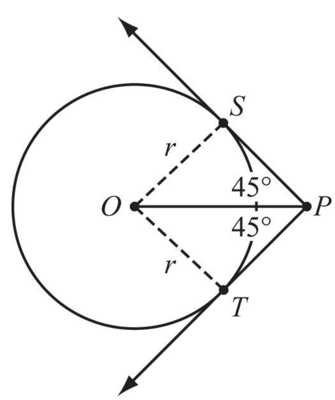
Triangle *PSO* is a 30-60-90 triangle. The length of the shorter leg is 6, so the length of $\overline{OP}$, the hypotenuse, is $2 \bullet 6 = 12$. The distance from *P* to the circle is $PO - 6 = 12 - 6 = 6$.

b. The figure below illustrates circle *O* and point *P* with the angles described in part b.



Triangle *PSO* is a 30-60-90 triangle. The length of the shorter leg is *r*, so the length of $\overline{OP}$, the hypotenuse, is 2*r*. The distance from *P* to the circle is $PO - r = 2r - r = r$.

c. The figure below illustrates circle *O* and point *P* with the angles described in part c.



Triangle *PSO* is a 45-45-90 triangle. The length of a leg is *r*, so the length of $\overline{OP}$, the hypotenuse, is $r\sqrt{2}$. The distance from *P* to the circle is $PO - r = r\sqrt{2} - r = r\left(\sqrt{2} - 1\right)$.

EOCT Practice Items

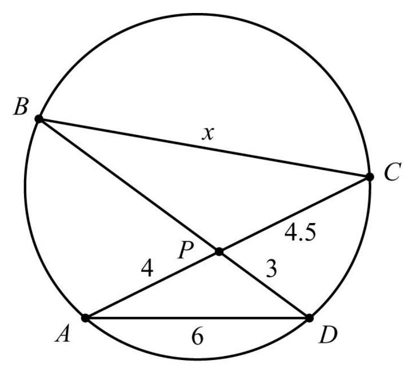
1) The center of circle *O* is located at $(3, 4)$ on the coordinate plane. The radius of circle *O* is $\sqrt{3}$ units. Point *P* is located at $(7, 0)$.

What is the length of $\overline{PT}$, the segment from point *P* that is tangent to circle *O* at point *T*?

A. $\sqrt{13}$ units
B. $\sqrt{19}$ units
C. $\sqrt{29}$ units
D. $\sqrt{35}$ units

[Key: C]

2) Points *A, B, C,* and *D* are on circle *P* as shown.

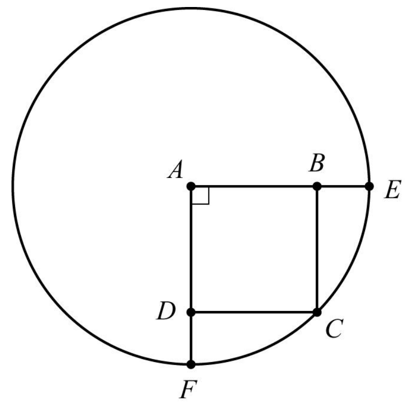


What is the value of *x*?

A. 7.5
B. 8
C. 9
D. 12

[Key: C]

3) Quadrilateral *ABCD* is a square in circle *A*.



If $\overline{AE} = 3\,\text{cm}$, what is the length of $\overline{DF}$?

A. $\sqrt{2}$

B. $3 - 3\sqrt{2}$

C. $\dfrac{3\sqrt{2}}{2}$

D. $3 - \dfrac{3\sqrt{2}}{2}$

[Key: D]

PROPERTIES OF CIRCLES: ARC LENGTHS AND SECTOR AREAS

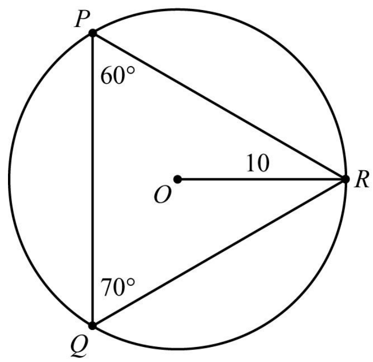
KEY IDEAS

1. The length of an arc that corresponds to a central angle with a measure of θ degrees in a circle with a radius r is $\dfrac{\theta}{360}$ times the circumference of the circle; i.e., $\dfrac{\theta}{360} \bullet 2\pi r = \dfrac{\theta \pi r}{180}$.

2. The area of a sector that corresponds to a central angle with a measure of θ degrees in a circle with a radius r is $\dfrac{\theta}{360}$ times the area of the circle; i.e., $\dfrac{\theta}{360} \bullet \pi r^2$.

REVIEW EXAMPLES

1) Triangle PQR is inscribed in circle O, as shown.



Find the lengths of the arcs $\overset{\frown}{PR}$, $\overset{\frown}{RQ}$, and $\overset{\frown}{QP}$.

Solution:

The measures of the arcs are twice the measures of the inscribed angles that correspond with each arc; i.e., $m\overset{\frown}{PR} = 2(70) = 140$, $m\overset{\frown}{RQ} = 2(60) = 120$, and $m\overset{\frown}{QP} = 2(50) = 100$.

The arc length of $\overset{\frown}{PR} = \dfrac{140}{360}(20\pi) = \dfrac{70\pi}{9}$.

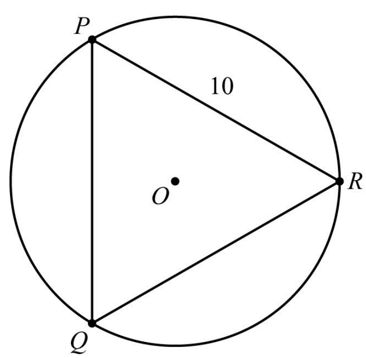
The arc length of $\overset{\frown}{RQ} = \dfrac{120}{360}(20\pi) = \dfrac{20\pi}{3}$.

The arc length of $\overset{\frown}{QP} = \dfrac{100}{360}(20\pi) = \dfrac{50\pi}{9}$.

2) An equilateral triangle that has sides 10 cm long is inscribed in a circle. What is the area of the circle?

Solution:

The figure below illustrates the triangle and the circle. The center of the circle is located at point O.

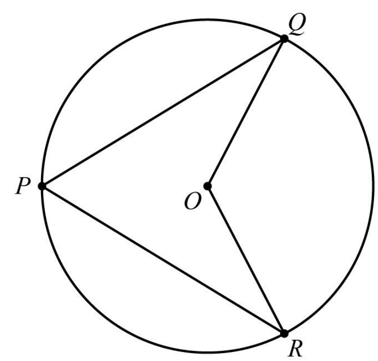


Since the sides of the triangle are all chords of circle *O*, point *O* is on the perpendicular bisectors of each of the sides of the triangle. The triangle is equilateral so the perpendicular bisectors are also the angle bisectors.

Let point *M* be the midpoint of side $\overline{PR}$. Triangle *MOR* is a 30-60-90 right triangle with a long leg that measures $\frac{10}{2} = 5$ cm. Radius $\overline{OR}$ is the hypotenuse of that triangle, so $OR = 2\left(\frac{5}{\sqrt{3}}\right) = \frac{10}{\sqrt{3}}$.

The area of circle *O* is $\pi r^2 = \pi\left(\frac{10}{\sqrt{3}}\right)^2 = \frac{100\pi}{3} = 33\frac{1}{3}\pi$.

3) In circle *O*, $\overline{PQ} \cong \overline{PR}$.



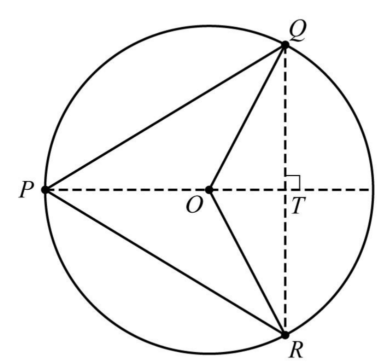
a. If $m\widehat{QR} = 120°$ and the radius of circle *O* is 20 cm, what is the area of sector *QOR?*

b. What is the area of quadrilateral *QPRO*?

Solution:

a. The area of sector *QOR* is $\frac{120}{360} \times \pi(20)^2 = \frac{400\pi}{3}$.

b. To find the area of the quadrilateral, start by drawing chord $\overline{QR}$ with a midpoint at point *T*. Note that $\overleftrightarrow{TO}$ is the perpendicular bisector of $\overline{QR}$ and that, since $\overline{PQ} \cong \overline{PR}$, point *P* is on $\overleftrightarrow{TO}$ as well.



$\overline{QT}$ is an altitude of triangle *PQO*, so by finding *QT* we can find the area of triangle *PQO*, which is half the area of quadrilateral *QPRO*.

Triangle *QOT* is a 30-60-90 triangle with a hypotenuse of 20 cm. $\overline{QT}$ is the longer leg of that triangle, so $QT = \frac{20}{2}\sqrt{3} = 10\sqrt{3}$.

The area of $\triangle QOP = \frac{1}{2}(PO)(QT) = \frac{1}{2}(20)(10\sqrt{3}) = 100\sqrt{3}$, so the area of quadrilateral $QPRO = 2(100\sqrt{3}) = 200\sqrt{3}$.

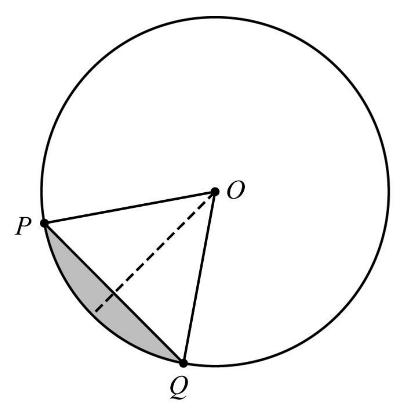
EOCT Practice Items

1) A circular pizza with a diameter of 15 inches is cut into 8 equal slices. What is the area of one slice?

A. 5.9 sq. in.
B. 22.1 sq in.
C. 88.4 sq. in.
D. 120 sq. in.

[Key: B]

2) In this diagram, triangle *OPQ* is equilateral, with vertex *O* at the center of a circle and vertices *P* and *Q* on the circle.



The radius of circle *O* is 12 cm. What is the area, in square units, of the shaded region?

A. $24\pi - 18$
B. $24\pi - 36\sqrt{3}$
C. $48\pi - 18$
D. $48\pi - 36\sqrt{3}$

[Key: B]

SURFACE AREA AND VOLUME OF SPHERES

KEY IDEAS

1. The surface area of a sphere with a radius r is given by $SA = 4\pi r^2$. The volume of the sphere is given by $V = \frac{4}{3}\pi r^3$.

2. Scale factors for linear dimensions, area, and volume state that if the ratio of the linear dimensions of two figures is $a : b$, then
   - the ratio of the areas is $a^2 : b^2$, and
   - the ratio of the volumes is $a^3 : b^3$.

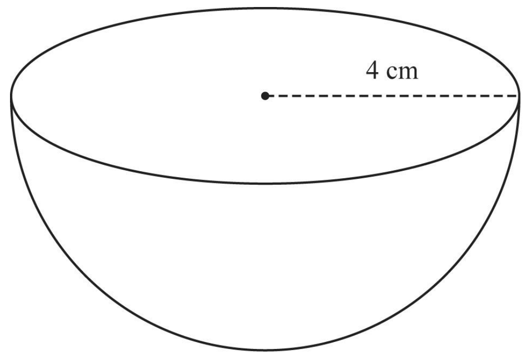
REVIEW EXAMPLES

1) A sphere has a diameter of 10 meters.
   a. What is the circumference of a great circle of the sphere?
   b. What is the surface area of the sphere?
   c. What is the volume of the sphere?
   d. If the diameter is decreased by a factor of 10, how will it affect the circumference, surface area, and volume?

Solution:

a. The circumference is 10π meters. Note that 10m is the **diameter**, not the radius, of the sphere.
b. The surface area is $4\pi r^2 = 4\pi(5)^2 = 100\pi$ square meters.
c. The volume is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(5)^3 = \frac{500\pi}{3}$ cubic meters.
d. The circumference will be decreased by a factor of 10, the surface area will be decreased by a factor of 100, and the volume will be reduced by a factor of 1000. [The resulting values would be π meters for the circumference, π square meters for the surface area, and $\frac{\pi}{6}$ cubic meters for the volume.]

2) The figure below represents half a sphere.



Find the surface area and volume of the figure.

Solution:

The surface area of ***half*** a sphere is $SA = \frac{\mathbf{1}}{\mathbf{2}} \times 4\pi r^2 = 2 \times \pi \times 4^2 = 32\pi = 100.53\,\text{cm}^2$

The volume of ***half*** a sphere is $V = \frac{\mathbf{1}}{\mathbf{2}} \times \frac{4}{3}\pi r^3 = \frac{2}{3} \times \pi \times 4^3 = 42.67\pi = 134.05\ \text{cm}^3$

EOCT Practice Items

1) A map company makes a globe in the shape of a sphere. The company plans to make a new model globe with a diameter that is 20% larger than the diameter of the original model.

By what percent will the surface area of the new model globe increase compared to the surface area of the original model?

A. 4%
B. 20%
C. 40%
D. 44%

[Key: D]

2) A sphere has a volume of 39π cubic centimeters. What is the surface area of the sphere?

A. 3.0π sq. cm
B. 9.5π sq. cm
C. 38.0π sq. cm
D. 81.5π sq. cm

[Key: C]

Unit 6: Exponential and Inverses

In this unit, students study exponential functions and how they apply to business and science, as well as identify critical points and their interpretations. Students learn to solve simple equations by changing bases. This unit also explores inverses of functions and uses compositions to verify that functions are inverses of each other.

KEY STANDARDS

MM2A2. Students will explore exponential functions.

a. Extend properties of exponents to include all integer exponents.
b. Investigate and explain characteristics of exponential functions, including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, rates of change, and end behavior.
c. Graph functions as transformations of $f(x) = a^x$.
d. Solve simple exponential equations and inequalities analytically, graphically, and by using appropriate technology.
e. Understand and use basic exponential functions as models of real phenomena.
f. Understand and recognize geometric sequences as exponential functions with domains that are whole numbers.
g. Interpret the constant ratio in a geometric sequence as the base of the associated exponential function.

MM2A5. Students will explore inverses of functions.

a. Discuss the characteristics of functions and their inverses, including one-to-oneness, domain, and range.
b. Determine inverses of linear, quadratic, and power functions and functions of the form $f(x) = \frac{a}{x}$, including the use of restricted domains.
c. Explore the graphs of functions and their inverses.
d. Use composition to verify that functions are inverses of each other.

EXPONENTIAL FUNCTIONS

KEY IDEAS

1. There are five basic ***properties of exponents***:

   a. $a^n a^m = a^{n+m}$

   Example:

   $2^3 \bullet 2^5 = 2^{3+5} = 2^8 = 256$

   b. $(a^n)^m = a^{n \bullet m}$

   Example:

   $(3^2)^3 = 3^{2 \bullet 3} = 3^6 = 729$

   c. $a^0 = 1$

   Example:

   $5^0 = 1$

   d. $\dfrac{a^n}{a^m} = a^{n-m}$

   Example:

   $\dfrac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$

   e. $a^{-n} = \dfrac{1}{a^n}$

   Example:

   $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

   Example:

   $\left(\dfrac{1}{4}\right)^{-2} = \left(\dfrac{4}{1}\right)^2 = 4^2 = 16$

Example:

$$\frac{2^3}{2^5} = 2^{3-5} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

2. The properties of exponents can be used to solve exponential equations. The first step is to rewrite the equation so that the bases on both sides of the equation are the same. If the bases on both sides are the same, then the exponents must be equal.

Example:

$$25^{x-2} = 125^{4x}$$
$$(5^2)^{x-2} = (5^3)^{4x}$$
$$5^{2(x-2)} = 5^{3(4x)}$$
$$5^{2x-4} = 5^{12x}$$

Now that the bases are the same, the exponents are equal.

$$2x - 4 = 12x$$
$$10x = -4$$
$$x = -\frac{2}{5}$$

3. An ***exponential function*** with a base b is written $f(x) = b^x$, where b is a positive number other than 1.

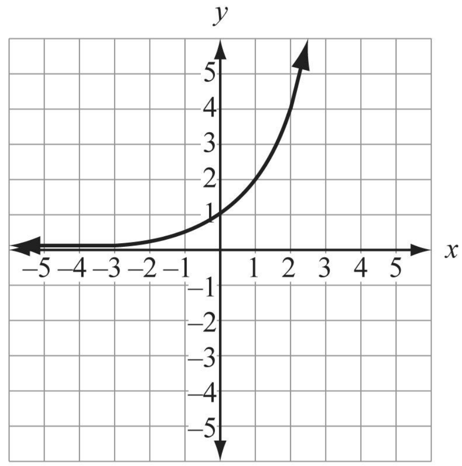
4. An ***exponential growth function*** can be written in the form $f(x) = ab^x$, where $a > 0$ and $b > 1$. An ***exponential decay function*** can be written in the form $f(x) = ab^x$, where $a > 0$ and $0 < b < 1$.

Example:

The function $f(x) = 2^x$ is an ***exponential growth function***. This is a table of values for the function.

x	$f(x)$
–3	—
–2	—
–1	—
0	1
1	2
2	4
3	8

This is the graph of the function.



Notice the ***end behavior*** of the graph. As the value of x increases, the graph moves up to the right, and the value of y increases without bound. As the value of x decreases, the graph moves down to the left and approaches the x-axis or $y = 0$, which is called an ***asymptote***.

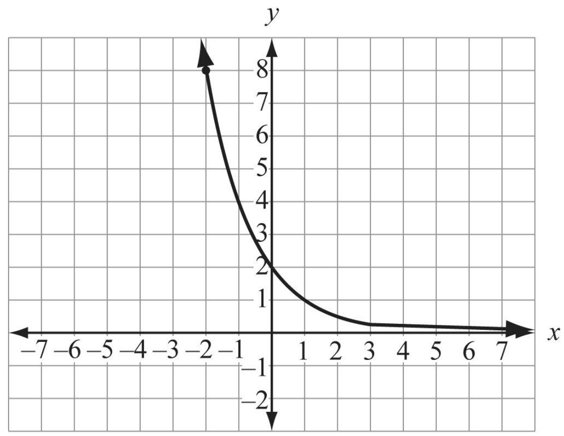
The ***domain*** for this function is all real numbers. The ***range*** is $y > 0$.

Example:

The function $f(x) = 2\left(\frac{1}{2}\right)^x$ is an ***exponential decay function***. This is a table of values for the function.

x	$f(x)$
–3	16
–2	8
–1	4
0	2
1	1
2	—
3	—

This is the graph of the function.



Notice the end behavior of the graph. As the value of x increases, the graph moves down to the right and approaches the x-axis or the asymptote $y = 0$. As the value of x decreases, the graph moves up to the left. As the value of x gets very small, the value of y increases without bound.

The domain for this function is all real numbers. The range is $y > 0$.

5. One, two, three, or all four of the transformations shown below can be applied to an exponential function.

a. The graph of $f(x) = b^x$ rewritten in the form $f(x) = ab^{x-h} + k$ is ***translated horizontally*** to the right h units if $h > 0$ or to the left h units if $h < 0$.

b. The graph of $f(x) = b^x$ rewritten in the form $f(x) = ab^{x-h} + k$ is ***translated vertically*** up k units if $k > 0$ or down k units if $k < 0$.

c. The graph of $f(x) = b^x$ rewritten in the form $f(x) = ab^{x-h} + k$ is ***vertically stretched*** if $a > 1$. The graph is ***vertically shrunk*** if $0 < a < 1$.

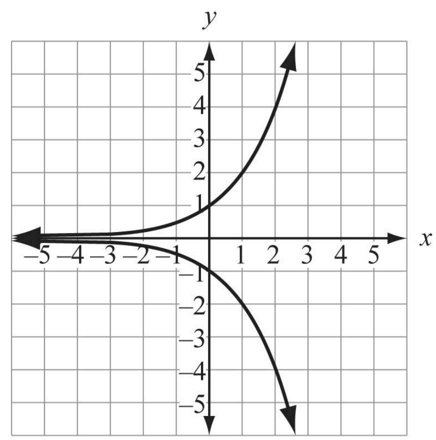
d. The graph of $f(x) = b^x$ is ***reflected across the x-axis*** if b^x is multiplied by -1. The graph of $f(x) = b^x$ is ***reflected across the y-axis*** if the exponent x in b^x is multiplied by -1.

Example:

The graph of $f(x)=2^x$ rewritten in the form $f(x)=3(2^{x-1})+6$ is translated horizontally to the right 1 unit, translated vertically up 6 units, and vertically stretched by a factor of 3.

Example:

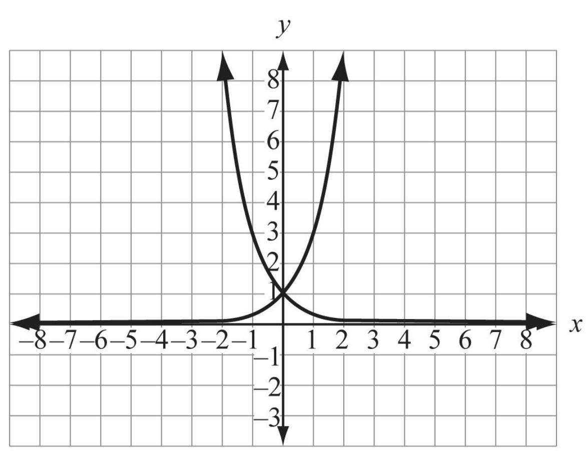
This coordinate grid shows the graphs of $f(x)=2^x$ and $f(x)=-2^x$.



Remember that -2^x is the same thing as $-(2^x)$.

Example:

This coordinate grid shows the graphs of $f(x)=3^x$ and $f(x)=3^{-x}$.



Remember that 3^{-x} can be rewritten as $\left(\frac{1}{3}\right)^x$ using the rule of exponents.

The function $f(x)=3^x$ is a growth function and $f(x)=3^{-x}$ is a decay function.

6. The graph of an exponential function written in the form $f(x) = ab^x$ passes through the point (0, a) and approaches the x-axis as an asymptote. To graph an exponential function in the form $f(x) = ab^{x-h} + k$, first sketch the graph of $f(x) = ab^x$ and then translate the graph horizontally by h units and vertically by k units.

7. A ***geometric sequence*** can be written as an exponential function with a domain that consists of positive integers. In a geometric sequence, the ratio of any term to its preceding term is constant. The ***constant ratio*** is called the ***common ratio*** and is the base of the associated exponential function. The nth term of a geometric sequence with the first term and with a common ratio r is $f(n) = ar^{n-1}$.

Example:

A geometric sequence is given as (2, 6, 18, 54, . . .). This sequence starts with the number 2. To get the next term in the sequence you always multiply by 3. This is the common ratio. The sequence is represented by the function $f(n) = 2(3)^{n-1}$.

The first three terms of the sequence are found using the function, as shown below.

First term:

$f(n) = 2(3)^{n-1}$

$f(1) = 2(3)^{1-1} = 2(3)^0 = 2(1) = 2$

Second term:

$f(n) = 2(3)^{n-1}$

$f(2) = 2(3)^{2-1} = 2(3)^1 = 2(3) = 6$

Third term:

$f(n) = 2(3)^{n-1}$

$f(3) = 2(3)^{3-1} = 2(3)^2 = 2(9) = 18$

8. The ***natural base*** e is an irrational number. It is defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. As n increases, the value of the expression $\left(1+\frac{1}{n}\right)^n$ approaches e, and $e \approx 2.718281828459$. The natural base e is used in the formula $A = Pe^{rt}$ to calculate continuously compounded interest.

REVIEW EXAMPLES

1) A city's population is 12,800. It is predicted that the population will increase by 3% each year.

   a. Write a function that can be used to find the city's population after *n* years.

   b. If the city's population, *p*, continues to grow 3% per year, what will the approximate population be in 15 years?

Solution:

a. $p(n) = 12{,}800 \bullet (1.03)^n$

b. $p(n) = 12{,}800 \bullet (1.03)^n$

$$p(15) = 12{,}800 \bullet (1.03)^{15}$$
$$p = 12{,}800 \bullet 1.557967417$$
$$p \approx 19{,}942$$

2) Deb bought a new boat for $20,000. She estimates that the value of the boat will decrease by 12% each year.

   a. Write an exponential decay function that represents the value of the boat after *t* years.

   b. Graph the exponential decay function from part a.

   c. What is the domain of the function?

   d. What is the range of the function?

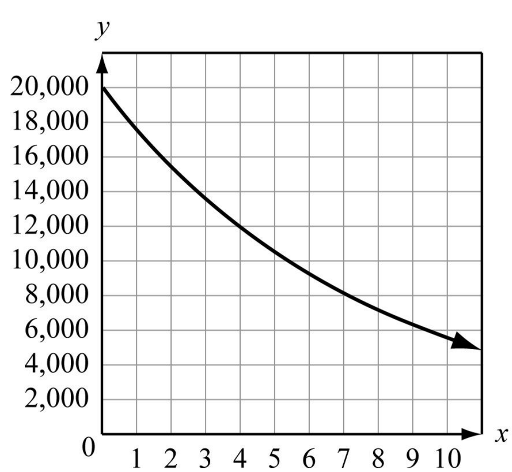
   e. What is the asymptote of the graph?

   f. After how many years will the boat have a value of about $4,900?

Solution:

a. $f(x) = 20{,}000(1 - 0.12)^t$

b.



c. all real numbers greater than or equal to 0

d. all values of y so that $0 < y \leq 20,000$

e. The asymptote is the x-axis.

f. 11 years

EOCT Practice Items

1) The function $f(x)$ has these properties.

- **As x increases, $f(x)$ approaches 3.**
- **As x decreases, $f(x)$ increases.**
- **The domain of $f(x)$ is all real numbers.**

Which of the following could be the function?

A. $f(x) = -2^{x-3}$

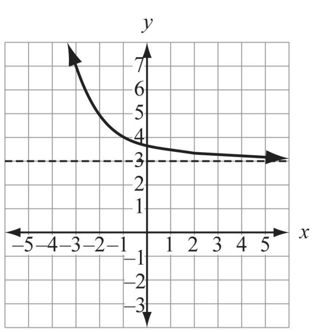
B. $f(x) = \left(\frac{1}{2}\right)^{x-3}$

C. $f(x) = -2^{x} + 3$

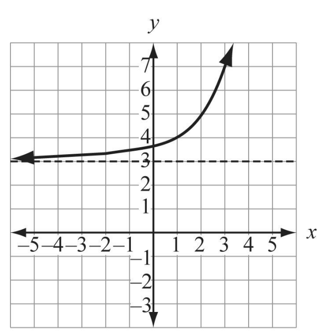
D. $f(x) = \left(\frac{1}{2}\right)^{x} + 3$

[Key: D]

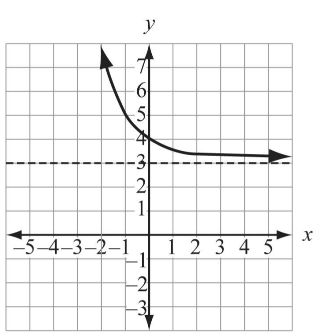
2) Which graph represents $f(x) = \left(\frac{1}{2}\right)^{x+1} + 3$?



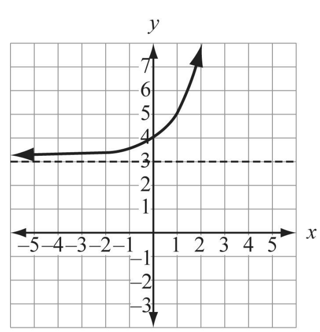
A.



B.



C.



D.

[Key: A]

COMPOSITION AND INVERSE FUNCTIONS

KEY IDEAS

1. The ***composition of functions*** is a method of combining functions. The composition of functions $f(x)$ and $g(x)$ is written $(f \circ g)(x)$ and is defined by $(f \circ g)(x) = f[g(x)]$.

Example:

Let $f(x) = x^2$ and $g(x) = 2x + 3$. Find $(f \circ g)(x)$.

$(f \circ g)(x) = f[g(x)]$

Substitute $2x + 3$ for $g(x)$ $\quad = f(2x + 3)$

It was given that $\quad f(x) = x^2$

So it follows that $\quad f(2x+3) = (2x+3)^2$

That simplifies to $\quad = 4x^2 + 12x + 9$

So $\quad (f \circ g)(x) = 4x^2 + 12x + 9$

Example:

Let $f(x) = x^2$ and $g(x) = 2x + 3$. Find $(g \circ f)(x)$.

$(g \circ f)(x) = g[f(x)]$

Substitute x^2 for f $\quad = g$

It was given that $\quad g(x) = 2x + 3$

So it follows that $\quad g(x^2) = 2(x^2) + 3$

That simplifies to $\quad = 2x^2 + 3$

So $\quad (g \circ f)(x) = 2x^2 + 3$

Notice that although the two examples are compositions of the same two functions, the order in which they are listed makes a difference in the outcome.

2. An ***inverse function*** reverses the process of the original function. A function maps the input values onto the output values. An inverse function maps the output values onto the original input values. Switching the x-values and the y-values in an input-output table produces an inverse function. The same thing is true if the x and the y variables are

switched in an equation. The inverse of a function is denoted by f^{-1} and is read "*f* inverse."

3. The input (domain) and output (range) values of all functions have a ***one-to-one*** relationship. For each input, there is exactly one output. If a function has an inverse that is also a function, then the input and output values of the inverse function must also have a one-to-one relationship. Therefore, a function that has an inverse that is a function is called a ***one-to-one function***.

   Tables of some values of the function $f(x)=x^3$ and its inverse are shown below. Notice that both tables show a one-to-one relationship. This one-to-one relationship is true for all values of both x and $f(x)$ for both the original function and its inverse. Therefore, the function $f(x)=x^3$ is a one-to-one function.

Example:

Original Function

x	$f(x)$
–2	–8
–1	–1
0	0
1	1
2	8

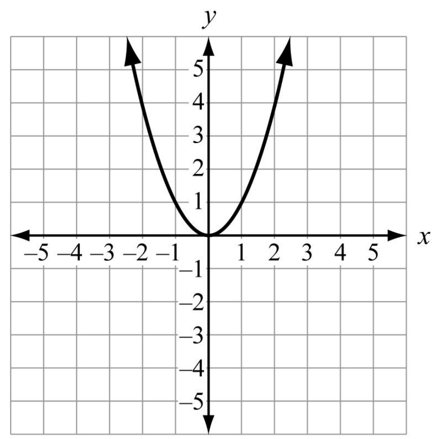
Inverse Function

x	$g(x)$
–8	–2
–1	–1
0	0
1	1
8	2

4. A function may or may not have an inverse function. You can use the ***horizontal line test*** on the graph of a function to determine if it has an inverse function. If there is no horizontal line that intersects the graph at more than one point, then it will have an inverse function. If the inverse maps an x value with more than one y-value, then it is not an inverse function. It can, however, be an inverse relation.

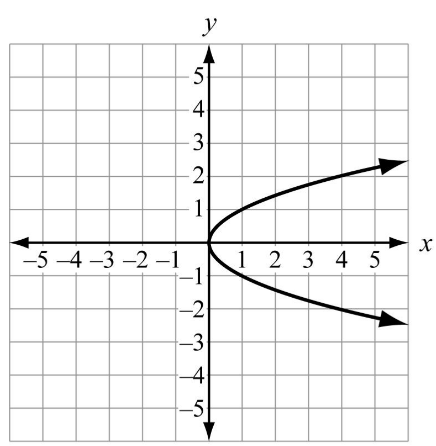
Example:

This is a graph of the function $f(x) = x^2$.

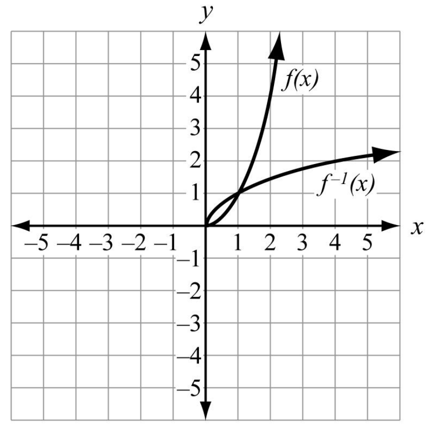


The function $f(x) = x^2$ is not a one-to-one function. There is at least one horizontal line that intersects the graph at more than one point. In other words, at least one y-value is paired with two different x-values, as in the case where a horizontal line passes through (−1, 1) and (1, 1).

The graph of the inverse of the relation $f(x) = x^2$ is clearly not a function since it does not pass the vertical line test of functions, as shown in the graph below. In other words, at least one x-value is paired with two different y-values, as in the case where a vertical line passes through (4, 2) and (4, −2).



5. A function that is not a one-to-one function when its domain is the set of all real numbers may become a one-to-one function if the domain is restricted appropriately. For example, the function $f(x) = x^2$ as shown in Key Idea #4 is not a one-to-one function. However, if the domain is restricted to only positive real numbers, it is a one-to-one function, as shown on the next page.



Important Tip

Notice that the graph of the inverse of the function is the same as the reflection of the function across the line $y = x$.

6. All linear functions *except* horizontal lines are one-to-one functions.

7. Another way to verify that two functions are inverses of each other is by using composition. The composition of the two functions must give the ***identity function*** which is $f(x) = x$.

Example:

Show that the two functions $f(x) = 2x + 3$ and $g(x) = \dfrac{x-3}{2}$ are inverses of each other.

Find f [g (x)].

$$\begin{aligned} f[g(x)] &= f\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right) + 3 \\ &= x - 3 + 3 \\ &= x \end{aligned}$$

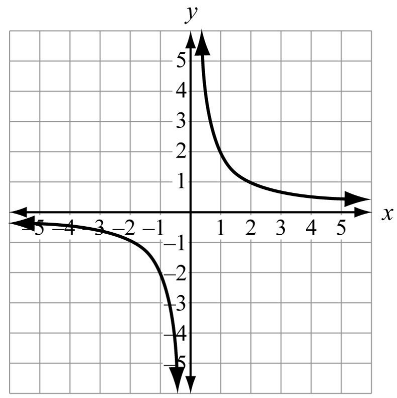
or

Find

$$\begin{aligned} g[f(x)] &= g(2x+3) \\ &= \frac{2x+3-3}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

Since $f[g(x)] = g[f(x)] = x,$ then the composition of the functions gives the identity function and the functions are inverses of each other. You only need to find one composition to verify that the functions are inverses of each other. However, it can be a good idea to find both compositions as a way of verifying that your work is correct.

8. The inverse of a function in the form $f(x)=\frac{a}{x}$ or $y=\frac{a}{x}$ is the same function. The graph of $f(x)=\frac{2}{x}$ is shown below.



If this graph is reflected across the line $y = x$, the result is the same graph.

This is the table for the function $f(x)=\frac{2}{x}$.

x	*f*(*x*)
−4	−−
−3	−−
−2	−1
−1	−2
0	undefined
1	2
2	1
3	—
4	—

If you swap the *x* values for the *y* values in the table and graph them, it is the same graph. Notice that the domain for the function is the set of all real numbers except 0. The function is undefined at $x = 0$.

REVIEW EXAMPLES

1) A table of values for the function $f(x)$ is shown below.

x	$f(x)$
−2	−7
−1	0
0	1
1	2
2	9
3	28

a. What is the table of values for the inverse of the function?

b. Is $f(x)$ a one-to-one function? Explain your reasoning.

Solution:

a.

x	$f(x)$
−7	−2
0	−1
1	0
2	1
9	2
28	3

b. It is a one-to-one function because the table of the inverse of the function also represents a function. For each input number there is exactly one output number.

2) Verify that $f(x)=\frac{1}{4}x^2, x\geq 0,$ and $g(x)=\sqrt{4x}$ are inverse functions.

Solution:

Find $f(g(x))$.

$$\begin{aligned} f(g(x)) &= f\left(\sqrt{4x}\right) \\ &= \frac{1}{4}\left(\sqrt{4x}\right)^2 \\ &= \frac{1}{4}(4x) \\ &= x \end{aligned}$$

Since the composition equals x, then the functions are inverses.

EOCT Practice Items

1) Use this function to answer the question.

$$f(x)=\frac{2}{x}+3$$

What value is NOT included in the domain of the inverse of this function?

A. 0
B. 1
C. 2
D. 3

[Key: D]

2) Use these functions to answer the question.

$$f(x)=4x-2$$
$$g(x)=\frac{x+2}{4}$$
$$f(g(x))=x$$

Which statement about the functions $f(x)$ and $g(x)$ is true?

A. They are inverse functions because $f(g(x))$ is not equal to 0.
B. They are inverse functions because $f(g(x))$ is equal to x.
C. They are not inverse functions because $f(g(x))$ is not equal to 0.
D. They are not inverse functions because $f(g(x))$ is equal to x.

[Key: B]

Appendix A
EOCT Sample Overall Study Plan Sheet

Here is a sample of what an OVERALL study plan might look like. You can use the Blank Overall Study Plan Sheet in Appendix B or create your own.

Materials/Resources I May Need When I Study:
(You can look back at page 6 for ideas.)

1. *This study guide*
2. *Pens/pencils*
3. *Highlighter*
4. *Notebook*
5. *Dictionary*
6. *Calculator*
7. *Mathematics textbook*

Possible Study Locations:

- First choice: *The library*
- Second choice: *My room*
- Third choice: *My mom's office*

Overall Study Goals:

1. *Read and work through the entire study guide.*
2. *Answer the sample questions and study the answers.*
3. *Do additional reading in a mathematics textbook.*

Number of Weeks I Will Study: *6 weeks*

Number of Days a Week I Will Study: *5 days a week*

Best Study Times for Me:

- Weekdays: *7:00 p.m. – 9:00 p.m.*
- Saturday: *9:00 a.m. – 11:00 a.m.*
- Sunday: *2:00 p.m. – 4:00 p.m.*

Appendix B
Blank Overall Study Plan Sheet

Materials/Resources I May Need When I Study:
(You can look back at page 6 for ideas.)

1. ___________________________________
2. ___________________________________
3. ___________________________________
4. ___________________________________
5. ___________________________________
6. ___________________________________

Possible Study Locations:

- First choice: ________________________________
- Second choice ________________________________
- Third choice ________________________________

Overall Study Goals:

1. ________________________________
2. ________________________________
3. ________________________________
4. ________________________________
5. ________________________________

Number of Weeks I Will Study: _________________________

Number of Days a Week I Will Study: ____________________

Best Study Times for Me:

- Weekdays: ______________________________________
- Saturday: ______________________________________
- Sunday: ____

Appendix C
EOCT Sample Daily Study Plan Sheet

Here is a sample of what a DAILY study plan might look like. You can use the Blank Daily Study Plan Sheet in Appendix D or create your own.

Materials I May Need Today:

1. *Study guide*
2. *Pens/pencils*
3. *Notebook*

Today's Study Location: *The desk in my room*

Study Time Today: *From 7:00 p.m. to 8:00 p.m. with a short break at 7:30 p.m.*
(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. "Doing time" at your desk doesn't count as real studying.)

If I Start to Get Tired or Lose Focus Today, I Will: *Do some sit-ups*

Today's Study Goals and Accomplishments: (Be specific. Include things like number of pages, units, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small "chunks" or blocks of material at a time.)

Study Task	*Completed*	*Needs More Work*	*Needs More Information*
1. Review what I learned last time	X		
2. Study the first main topic in Unit 1	X		
3. Study the second main topic in Unit 1		X	

What I Learned Today:

1. *Reviewed horizontal and vertical shifts*
2. *Locating the vertex of a quadratic function*
3. *Reviewed the definitions of an arithmetic series*

Today's Reward for Meeting My Study Goals: *Eating some popcorn*

Appendix D
Blank Daily Study Plan Sheet

Materials I May Need Today:

1. ______________________________
2. ______________________________
3. ______________________________
4. ______________________________
5. ______________________________

Today's Study Location: ____________________

Study Time Today: ____________________

(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. "Doing time" at your desk doesn't count for real studying.)

If I Start To Get Tired or Lose Focus Today, I Will: ______________________________

Today's Study Goals and Accomplishments: (Be specific. Include things like number of pages, sections, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small "chunks" or blocks of material at a time.)

Study Task	*Completed*	*Needs More Work*	*Needs More Information*
1.			
2.			
3.			
4.			
5.			

What I Learned Today:

1. ____
2. ____
3. ____

Today's Reward for Meeting My Study Goals: ________________________________