GEORGIA'S K-12
MATHEMATICS STANDARDS
2021

Linear Algebra with Computer Science Applications

MATHEMATICS
KEY COMPETENCIES & COURSE STANDARDS

WITH
LEARNING OBJECTIVES
IN PROGRESSION ORDER

Governor Kemp and Superintendent Woods are committed to the best set of academic standards for Georgia's students – laying a strong foundation of the fundamentals, ensuring age- and developmentally appropriate concepts and content, providing instructional supports to set our teachers up for success, protecting and affirming local control and flexibility regarding the use of mathematical strategies and methods, and preparing students for life. These Georgia-owned and Georgia-grown standards leverage the insight, expertise, experience, and efforts of thousands of Georgians to deliver the very best educational experience for Georgia's 1.7 million students.

In August 2019, Governor Brian Kemp and State School Superintendent Richard Woods announced the review and revision of Georgia's K-12 mathematics standards. Georgians have been engaged throughout the standards review and revision process through public surveys and working groups. In addition to educator working groups, surveys, and the Academic Review Committee, Governor Kemp announced a new way for Georgians to provide input on the standards: the Citizens Review Committee, a group composed of students, parents, business and community leaders, and concerned citizens from across the state. Together, these efforts were undertaken to ensure Georgians will have buy-in and faith in the process and product.

The Citizens Review Committee provided a charge and recommendations to the working groups of educators who came together to craft the standards, ensuring the result would be usable and friendly for parents and students in addition to educators. More than 14,000 Georgians participated in the state's public survey from July through September 2019, providing additional feedback for educators to review. The process of writing the standards involved more than 200 mathematics educators -- from beginning to veteran teachers, representing rural, suburban, and metro areas of our state.

Grade-level teams of mathematics teachers engaged in deep discussions; analyzed stakeholder feedback; reviewed every single standard, concept, and skill; and provided draft recommendations. To support fellow mathematics teachers, they also developed learning progressions to show when key concepts were introduced and how they progressed across grade levels, provided examples, and defined age/developmentally appropriate expectations.

These teachers reinforced that strategies and methods for solving mathematical problems are classroom decisions -- not state decisions -- and should be made with the best interest of the individual child in mind. These recommended revisions have been shared with the Academic Review Committee, which is composed of postsecondary partners, age/development experts, and business leaders, as well as the Citizens Review Committee, for final input and feedback.

Based on the recommendation of Superintendent Woods, the State Board of Education will vote to post the draft K-12 mathematics standards for public comment. Following public comment, the standards will be recommended for adoption, followed by a year of teacher training and professional learning prior to implementation.

Linear Algebra with Computer Science Applications

Overview

This document contains a draft of Georgia's 2021 K-12 Mathematics Standards for the High School Linear Algebra with Computer Science Applications Course, which is a fourth mathematics course option in the high school course sequence.

The standards are organized into big ideas, course competencies/standards, and learning objectives/expectations. The grade level key competencies represent the standard expectation of learning for students in each grade level. The competencies/standards are each followed by more detailed learning objectives that further explain the expectations for learning in the specific grade levels.

New instructional supports are included, such as clarification of language and expectations, as well as detailed examples. These have been provided for teaching professionals and stakeholders through the Evidence of Student Learning Column that accompanies each learning objective.

Course Description:

Linear Algebra with Computer Science Applications is designed to meet the needs of advanced students who have completed GSE Pre-Calculus or Accelerated GSE Pre-Calculus or the equivalent and will pursue careers which require linear algebra topics often associated with modern computer science.

The course will examine the use of vectors and matrices in mathematics and apply these concepts to computer science. There will be a strong focus on the presentation of mathematical ideas through both writing and programming. Mathematical concepts, such as vector spaces and Markov chains, will be presented through an abstract approach that characterizes upper-level mathematics courses. The goal is to give students the skills and techniques they will need as they study advanced mathematics or computer science at the college level. This is an alternative course for those students who do not wish to enroll in an Advanced Placement course, but who still wish to learn higher-level mathematics.

Course Description:

This course is designed for students who have successfully completed *GSE Pre-Calculus or Accelerated GSE Pre-Calculus*.

Georgia's K-12 Mathematics Standards - 2021
Mathematics Big Ideas and Learning Progressions, High School

Mathematics Big Ideas, HS

HIGH SCHOOL
MATHEMATICAL PRACTICES (MP)
MATHEMATICAL MODELING (MM)
NUMERICAL (QUANTITATIVE) REASONING (NR)
PATTERNING & ALGEBRAIC REASONING (PAR)
FUNCTIONAL & GRAPHICAL REASONING (FGR)
GEOMETRIC & SPATIAL REASONING (GSR)
DATA & STATISTICAL REASONING (DSR)
PROBABILISTIC REASONING (PR)
ABSTRACT & DIGITAL REASONING (ADR)

The 8 Mathematical Practices and the Mathematical Modeling Framework are essential to the implementation of the content standards presented in this course. More details related to these concepts can be found in the links below and in the first two standards presented in this course:

Mathematical Practices

Mathematical Modeling Framework

Linear Algebra with Computer Science Applications

The nine course standards listed below are the key content competencies students will be expected to master in this course. Additional clarity and details are provided through the classroom-level learning objectives and evidence of student learning details for each course standard found on subsequent pages of this document.

COURSE STANDARDS
LACS.MP: Display perseverance and patience in problem-solving. Demonstrate skills and strategies needed to succeed in mathematics, including critical thinking, reasoning, and effective collaboration and expression. Seek help and apply feedback. Set and monitor goals.
LACS.MM.1: Apply mathematics to real-life situations; model real-life phenomena using mathematics.
LACS.ADR.2: Investigate and describe real-life problems in linear algebra using an object-oriented programming language.
LACS.GSR.3: Solve contextual, mathematical problems involving vectors to explain real-life phenomena.
LACS.PAR.4: Solve contextual, mathematical problems involving matrices to explain real-life phenomena.
LACS.GSR.5: Solve contextual, mathematical problems involving matrices as geometric transformations and to explain real-life phenomena.
LACS.PR.6: Using probabilistic and quantitative reasoning, solve contextual, mathematical problems using Markov chains to explain real-life phenomena.
LACS.PAR.7: Solve contextual, mathematical problems using vector spaces to explain real-life phenomena.
LACS.PAR.8: Solve contextual, mathematical problems using eigenvalues and eigenvectors to explain real-life phenomena.

Linear Algebra with Computer Science Applications

MATHEMATICAL MODELING		
LACS.MM.1: Apply mathematics to real-life situations; model real-life phenomena using mathematics.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
LACS.MM.1.1	Explain contextual, mathematical problems using a mathematical model.	***Fundamentals*** • Students should be provided with opportunities to learn mathematics in the context of real-life problems. • Contextual, mathematical problems are mathematical problems presented in context where the context makes sense, realistically and mathematically, and allows for students to make decisions about how to solve the problem (model with mathematics).
LACS.MM.1.2	Create mathematical models to explain phenomena that exist in the natural sciences, social sciences, liberal arts, fine and performing arts, and/or humanities contexts.	***Fundamentals*** • Students should be able to use the content learned in this course to create a mathematical model to explain real-life phenomena.
LACS.MM.1.3	Using abstract and quantitative reasoning, make decisions about information and data from a contextual situation.	
LACS.MM.1.4	Use various mathematical representations and structures with this information to represent and solve real-life problems.	

***ABSTRACT & DIGITAL REASONING** – Programming*		
LACS.ADR.2: Investigate and describe real-life problems in linear algebra using an object-oriented programming language.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Use an object-oriented programming language to complete computer programming tasks.		
LACS.ADR.2.1	Utilize sets, lists, dictionaries, indexing, and tuples in programming languages.	
LACS.ADR.2.2	Show and explain how to program and apply modules and control statements in programming languages.	***Examples*** • Loops and conditionals such as if-then-else, for, and while-do
LACS.ADR.2.3	Program input and output features to read from and write to files in a programming assignment.	

***GEOMETRIC & SPATIAL REASONING** – Vectors*		
LACS.GSR.3: Solve contextual, mathematical problems involving vectors to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Use vectors to find and interpret geometric relationships.		
LACS.GSR.3.1	Use coordinates to represent points in *n* dimensions and define and use arithmetic operations on *n*-dimensional points.	
LACS.GSR.3.2	Use vectors to find and interpret geometrical relationships between points in two and three dimensions, such as distance, and generalize these relationships to higher dimensions using *n*-dimensional vectors.	***Fundamentals*** • Students should be provided opportunities to demonstrate understanding of geometric interpretations of vectors.
LACS.GSR.3.3	Interpret adding, scaling, and linear combinations of vectors geometrically and algebraically.	***Fundamentals*** • Students should be provided opportunities to demonstrate understanding of geometric interpretations of vectors.
Use the dot product and the cross product of two vectors to find and interpret geometric relationships.		
LACS.GSR.3.4	Find and use the dot product of two *n*-dimensional vectors.	***Examples*** • Determine whether two vectors are orthogonal using the dot product; find the projection of a vector along another; and find the angle between two vectors.
LACS.GSR.3.5	Use properties of the dot product to prove statements about vectors and to solve problems in context.	***Examples*** • Prove that the dot product of a vector with itself is equal to the square of the vector's magnitude. ***Relevance and Application*** • Find the angle between two vectors.
LACS.GSR.3.6	Use the triangle inequality in *n*-dimensions.	
LACS.GSR.3.7	Find and use the cross product of two 3-dimensional vectors.	***Examples*** • Given two vectors, find a vector orthogonal to the two vectors; find the areas of triangles and parallelograms.
Solve applied problems involving vectors using programming.		
LACS.GSR.3.8	Represent and perform vector operations using programming language classes that define the use of vectors.	***Examples*** • Represent a vector as an array of real numbers
LACS.GSR.3.9	Apply perfect secrecy, all-or-nothing secret sharing, and solving lights out games to vectors over GF(2).	
LACS.GSR.3.10	Use vector operations to program simple authentication schemes.	

***PATTERNING & ALGEBRAIC REASONING** – Matrices*		
LACS.PAR.4: Solve contextual, mathematical problems involving matrices to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Use matrices to solve systems of linear equations.		
LACS.PAR.4.1	Represent a linear system of three equations in three variables as an augmented matrix and reduce the matrix to row-echelon form.	
LACS.PAR.4.2	Interpret the nature of the solution of a system from its row-echelon form, and if there are infinitely many solutions, express them as a vector equation.	
LACS.PAR.4.3	Determine whether a vector is a linear combination of other given vectors; find the linear combination of vectors that results in a given vector.	***Fundamentals*** • Students should be given opportunities to interpret vectors written in **i**-**j**-**k** form as a linear combination the vectors **i**, **j**, and **k**.
LACS.PAR.4.4	Interpret linear dependence of vectors geometrically.	
LACS.PAR.4.5	Find the kernel of a matrix and explore the relationship between the kernel, the orthogonality of the vectors in the kernel, and the linear dependence of the rows/columns.	
Operations on and with matrices.		
LACS.PAR.4.6	Add two matrices, multiply a matrix by a scalar, find the transpose of a matrix.	
LACS.PAR.4.7	Determine when matrix multiplication is defined, and if defined, multiply two matrices by considering the matrix product as a dot product of a group of vectors.	
LACS.PAR.4.8	Determine when the inverse of a square matrix exists, and if it exists, find it by augmenting the identity matrix to the matrix and then use row operations.	
LACS.PAR.4.9	Decompose a matrix into its symmetric and skew-symmetric parts; decompose a matrix into its LU factorization.	
LACS.PAR.4.10	Solve a matrix equation using inverses; find all solutions to a matrix equation given one solution and the kernel.	
Solve applied problems involving matrices using programming.		
LACS.PAR.4.11	Improve the simple authentication scheme over GF(2).	
LACS.PAR.4.12	Show and explain how threshold secret sharing works in conjunction with Gaussian elimination through programming.	
LACS.PAR.4.13	Write code utilizing error-correcting concepts.	***Example*** • Hamming's code

GEOMETRIC & SPATIAL REASONING – *Matrices*		
LACS.GSR.5: Solve contextual, mathematical problems involving matrices as geometric transformations and to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** (not all inclusive; see Course Overview for more details)
Find and use matrices which represent geometric transformations.		
LACS.GSR.5.1	Given a 2-by-2 or 3-by-3 linear transformation matrix, describe the transformation a geometric figure undergoes.	***Fundamentals*** • Students should be provided opportunities to demonstrate understanding of geometric interpretations of matrices.
LACS.GSR.5.2	Find matrices that represent scalings, reflections, and rotations of geometric figures.	
LACS.GSR.5.3	Find a matrix that represents a combination of transformations.	
LACS.GSR.5.4	Find the image of a point under a transformation.	
LACS.GSR.5.5	Find the area of a polygon given its coordinates using matrices; find the area of the image of a polygon after a transformation.	
LACS.GSR.5.6	Write code to perform transformations in two-dimensional geometry using matrix operations.	
Use matrices as functions.		
LACS.GSR.5.7	Define functions from *n* dimensions to *m* dimensions as vectors and/or matrices.	
LACS.GSR.5.8	Find the image and preimage of a linear map using matrices; determine whether the linear map is one-to-one.	
LACS.GSR.5.9	Find and interpret geometrically the set of preimages of a vector under a given matrix.	

PROBABILISTIC REASONING – *Markov Chains*		
LACS.PR.6: Using probabilistic and quantitative reasoning, solve contextual, mathematical problems using Markov chains to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** (not all inclusive; see Course Overview for more details)
LACS.PR.6.1	Model a finite random process using transition matrices in a Markov chain.	
LACS.PR.6.2	Simulate the different stages of a Markov chain using random numbers.	

LACS.PR.6.3	Use matrix algebra to calculate the probability of future states of a Markov chain.	
LACS.PR.6.4	Determine the attractor for a regular Markov chain.	
LACS.PR.6.5	Use transition matrices to identify absorbing states of a Markov chain.	
LACS.PR.6.6	Apply Markov chains in context.	***Example*** • Google's PageRank algorithm, population movement, dance patterns, literary documents.
LACS.PR.6.7	Write a program to model the probabilities of real-life phenomena using a Markov chain.	

PATTERNING & ALGEBRAIC REASONING – *Vector Spaces*		
LACS.PAR.7: Solve contextual, mathematical problems using vector spaces to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Use properties of vector spaces to solve problems.		
LACS.PAR.7.1	Determine whether a given set of vectors generates a vector space.	
LACS.PAR.7.2	Justify whether a subset of a vector space is a subspace.	
LACS.PAR.7.3	Determine whether a given vector is in the linear span of a set of vectors.	
LACS.PAR.7.4	Determine whether two vector subspaces are orthogonal; find the orthogonal component of a given subspace.	
Find and use vectors as a basis.		
LACS.PAR.7.5	Determine whether a set of vectors is a basis for a vector space.	
LACS.PAR.7.6	Find the dimension of a vector space; find the dimensions of the row space, column space, and kernel for a given matrix; find the rank of a matrix.	
LACS.PAR.7.7	Find a matrix representing a linear map.	
LACS.PAR.7.8	Determine the change of representation for a linear transformation given two different bases on a vector space.	
LACS.PAR.7.9	Determine if two matrices are similar; determine if two matrices are orthogonal.	
LACS.PAR.7.10	Find an orthogonal basis for a given basis or subspace by applying the Gram-Schmidt orthonormalization process.	
LACS.PAR.7.11	Perform QR factorization of a matrix to solve matrix equations.	
Apply vector spaces and vectors as a basis in programming to solve contextual problems.		
LACS.PAR.7.12	Apply the method of least squares to find the line or parabola of best fit to approximate data in context.	

LACS.PAR.7.13	Apply the grow-and-shrink algorithm in the minimum spanning forest problem in GF(2).	
LACS.PAR.7.14	Apply the Exchange Lemma to image perspective rendering.	
LACS.PAR.7.15	Use bases to represent images and sounds as wavelets; perform wavelet transformation, implementation, and decomposition through programming.	
LACS.PAR.7.16	Program a Fast Fourier Transform to store a sequence of amplitude samples.	
LACS.PAR.7.17	Apply the Rank Theorem to demonstrate the simple authentication scheme.	

***PATTERNING & ALGEBRAIC REASONING** – Eigenvalues and Eigenvectors*		
LACS.PAR.8: Solve contextual, mathematical problems using eigenvalues and eigenvectors to explain real-life phenomena.		
Expectations		**Evidence of Student Learning** **(not all inclusive; see Course Overview for more details)**
Use and apply determinants.		
LACS.PAR.8.1	Evaluate the determinant of a matrix along any row or column and use a recursive procedure for evaluating a determinant for matrices larger than 3-by-3.	
LACS.PAR.8.2	Justify properties of the determinant.	***Example*** • Scalar multiplication; multiplying a row or a column by a scalar; adding a row to another row; adding a column to another column.
LACS.PAR.8.3	Calculate the determinant of the product of two matrices; calculate the determinant of the transpose of a matrix.	
LACS.PAR.8.4	Determine if a matrix has a nonzero determinant and extend the nonzero determinant property to problems involving linear dependency, rank, and matrix inverses.	
LACS.PAR.8.5	Extend the definition and geometric interpretation of the cross product to n – 1 vectors in n dimensions.	
LACS.PAR.8.6	Use Cramer's Rule to solve a system of linear equations.	***Strategies and Methods*** • For 2-by-2 and 3-by-3 matrices, use Cramer's Rule by hand. For larger than 3-by-3 matrices, use calculators, software, or student-written code.

Find and apply eigenvalues and eigenvectors.		
LACS.PAR.8.7	Find the characteristic polynomial of a matrix and interpret the characteristic polynomial geometrically.	
LACS.PAR.8.8	Find the eigenvalues and eigenvectors of a matrix and interpret them geometrically.	
LACS.PAR.8.9	Use a basis of eigenvectors to create a change of basis matrix.	
LACS.PAR.8.10	Find the dimension of the eigenspace corresponding to the eigenvalues of a symmetric matrix.	
LACS.PAR.8.11	Determine an orthogonal matrix that diagonalizes a given matrix.	
LACS.PAR.8.12	Apply eigenvalues and eigenvectors to problems in context.	***Example*** • Calculations in probability theory and dynamical systems; program face-recognition software (eigenfaces).

ESSENTIAL INSTRUCTIONAL GUIDANCE

MATHEMATICAL PRACTICES

The Mathematical Practices describe the reasoning behaviors students should develop as they build an understanding of mathematics – the “habits of mind” that help students become mathematical thinkers. There are eight standards, which apply to all grade levels and conceptual categories.

These mathematical practices describe how students should engage with the mathematics content for their grade level. Developing these habits of mind builds students’ capacity to become mathematical thinkers. These practices can be applied individually or together in mathematics lessons, and no particular order is required. In well-designed lessons, there are often two or more Standards for Mathematical Practice present.

Mathematical Practices	
LACS.MP: Display perseverance and patience in problem-solving. Demonstrate skills and strategies needed to succeed in mathematics, including critical thinking, reasoning, and effective collaboration and expression. Seek help and apply feedback. Set and monitor goals.	
Code	**Expectation**
LACS.MP.1	Make sense of problems and persevere in solving them.
LACS.MP.2	Reason abstractly and quantitatively.
LACS.MP.3	Construct viable arguments and critique the reasoning of others.
LACS.MP.4	Model with mathematics.
LACS.MP.5	Use appropriate tools strategically.
LACS.MP.6	Attend to precision.
LACS.MP.7	Look for and make use of structure.
LACS.MP.8	Look for and express regularity in repeated reasoning.

MATHEMATICAL MODELING

Teaching students to model with mathematics is engaging, builds confidence and competence, and gives students the opportunity to collaborate and make sense of the world around them, the main reason for doing mathematics. For these reasons, mathematical modeling should be incorporated at every level of a student's education. This is important not only to develop a deep understanding of mathematics itself, but more importantly to give students the tools they need to make sense of the world around them. Students who engage in mathematical modeling will not only be prepared for their chosen career but will also learn to make informed daily life decisions based on data and the models they create.

The diagram below is a mathematical modeling framework depicting a cycle of how students can engage in mathematical modeling when solving a real-life problem or task.

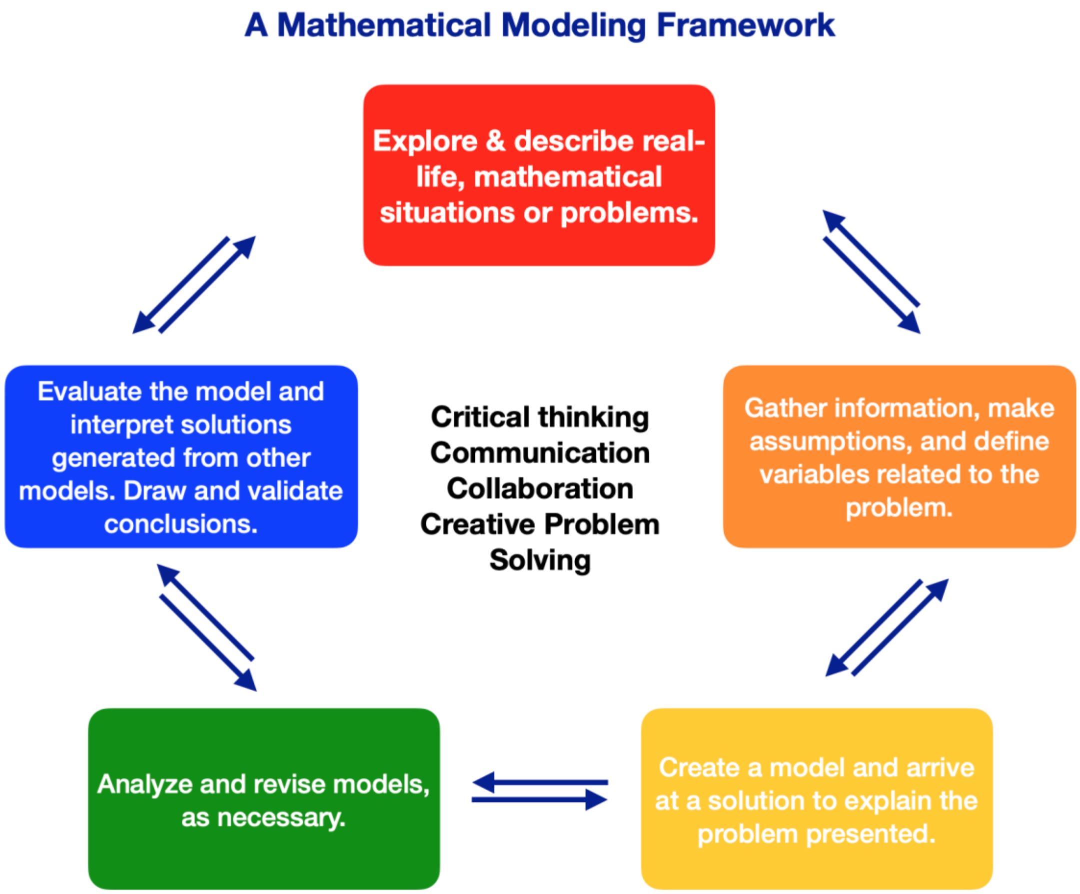

Image adapted from: Suh, Matson, Seshaiyer, 2017

FRAMEWORK FOR STATISTICAL REASONING

Statistical reasoning is important for learners to engage as citizens and professionals in a world that continues to change and evolve. Humans are naturally curious beings and statistics is a language that can be used to better answer questions about personal choices and/or make sense of naturally occurring phenomena. Statistics is a way to ask questions, explore, and make sense of the world around us.

The Framework for Statistical Reasoning should be used in all grade levels and courses to guide learners through the sense-making process, ultimately leading to the goal of statistical literacy in all grade levels and courses. Reasoning with statistics provides a context that necessitates the learning and application of a variety of mathematical concepts.

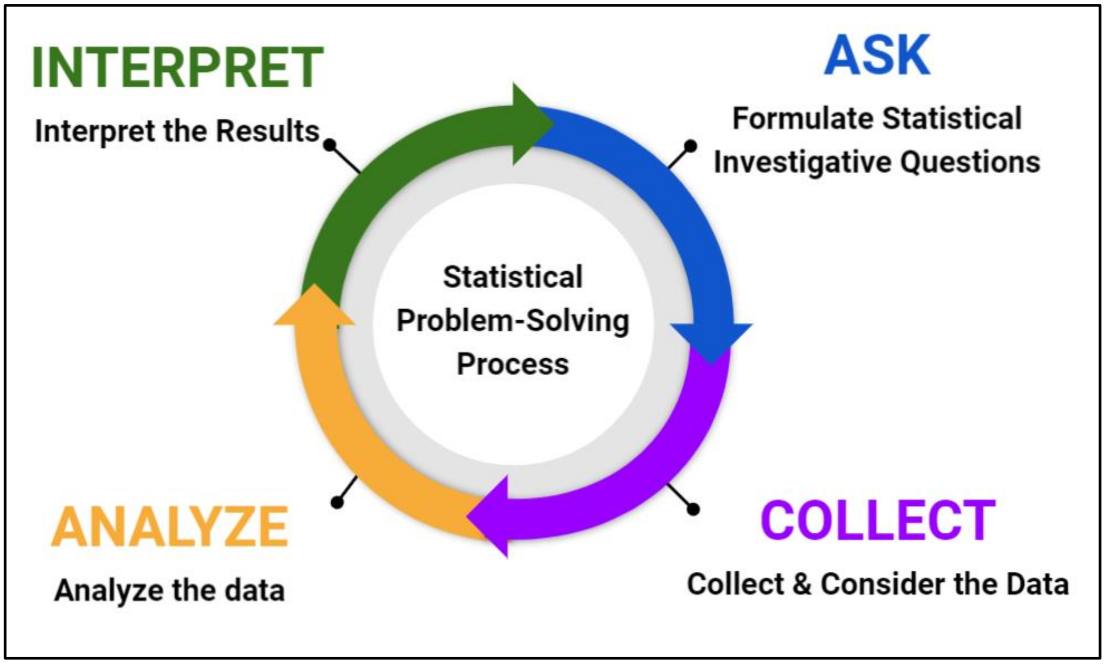

Figure 1: Georgia Framework for Statistical Reasoning

The following four-step statistical problem-solving process can be used throughout each grade level and course to help learners develop a solid foundation in statistical reasoning and literacy:

I. **Formulate Statistical Investigative Questions**
Ask questions that anticipate variability.

II. **Collect & Consider the Data**
Ensure that data collection designs acknowledge variability.

III. **Analyze the Data**
Make sense of data and communicate what the data mean using pictures (graphs) and words. Give an accounting of variability, as appropriate.

IV. **Interpret the Results**
Answer statistical investigative questions based on the collected data.