COORDINATE ALGEBRA

Study Guide

Georgia End-Of-Course Tests

January 2014

Revision Log

January 2014:

Unit 1 (Relationships Between Quantities)
page 22: Practice Item #2 replaced

Unit 4 (Describing Data)
pages 138-139: strikethrough text regarding median-median lines removed

Unit 6 (Connecting Algebra and Geometry Through Coordinates)
pages 172–173: Review Example #4 replaced

TABLE OF CONTENTS

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INTRODUCTION

This study guide is designed to help students prepare to take the Georgia End-of-Course Test (EOCT) for ***Coordinate Algebra***. This study guide provides information about the EOCT, tips on how to prepare for it, and some suggested strategies students can use to perform their best.

What is the EOCT? The EOCT program was created to improve student achievement through effective instruction and assessment of the material in the state-mandated content standards. The EOCT program helps ensure that all Georgia students have access to rigorous courses that meet high academic expectations. The purpose of the EOCT is to provide diagnostic data that can be used to enhance the effectiveness of schools' instructional programs.

The Georgia End-of-Course Testing program is a result of the A+ Educational Reform Act of 2000, O.C.G.A. §20-2-281. This act requires the Georgia Department of Education to create end-of-course assessments for students in grades nine through twelve for the following core high school subjects:

Mathematics

- Mathematics II: Geometry/Algebra II/Statistics

--OR--

- GPS Geometry

--OR--

- Coordinate Algebra (beginning 2012–2013)
- Analytic Geometry (beginning 2013–2014)

Social Studies

- United States History
- Economics/Business/Free Enterprise

Science

- Biology
- Physical Science

English Language Arts

- Ninth Grade Literature and Composition
- American Literature and Composition

Getting started: The HOW TO USE THE STUDY GUIDE section on page 6 outlines the contents in each section, lists the materials you should have available as you study for the EOCT, and suggests some steps for preparing for the ***Coordinate Algebra*** **EOCT**.

HOW TO USE THE STUDY GUIDE

This study guide is designed to help you prepare to take the ***Coordinate Algebra*** **EOCT**. It will give you valuable information about the EOCT, explain how to prepare to take the EOCT, and provide some opportunities to practice for the EOCT. The study guide is organized into three sections. Each section focuses on a different aspect of the EOCT.

The OVERVIEW OF THE EOCT section on page 8 gives information about the test: dates, time, question format, and number of questions that will be on the ***Coordinate Algebra*** **EOCT**. This information can help you better understand the testing situation and what you will be asked to do.

The PREPARING FOR THE EOCT section that begins on page 9 provides helpful information on study skills and general test-taking skills and strategies. It explains how to prepare before taking the test and what to do during the test to ensure the best test-taking situation possible.

The TEST CONTENT section that begins on page 15 explains what the ***Coordinate Algebra*** **EOCT** specifically measures. When you know the test content and how you will be asked to demonstrate your knowledge, it will help you be better prepared for the EOCT. This section also contains some sample EOCT test questions, helpful for gaining an understanding of how a standard may be tested.

With some time, determination, and guided preparation, you will be better prepared to take the ***Coordinate Algebra*** **EOCT**.

GET IT TOGETHER

In order to make the most of this study guide, you should have the following:

Materials:

* This study guide
* Pen or pencil
* Highlighter
* Paper

Resources:

* Classroom notes
* Mathematics textbook
* A teacher or other

Study Space:

* Comfortable (but not too comfortable)
* Good lighting
* Minimal distractions
* Enough work space

Time Commitment:

* When are you going to study?
* How long are you going to study?

Determination:

* Willingness to improve
* Plan for meeting

SUGGESTED STEPS FOR USING THIS STUDY GUIDE

1 Familiarize yourself with the structure and purpose of the study guide. (You should have already read the INTRODUCTION and HOW TO USE THE STUDY GUIDE. Take a few minutes to look through the rest of the study guide to become familiar with how it is arranged.)

2 Learn about the test and expectations of performance.
(Read OVERVIEW OF THE EOCT.)

3 Improve your study skills and test-taking strategies.
(Read PREPARING FOR THE EOCT.)

4 Learn what the test will assess by studying each unit and the strategies for answering questions that assess the standards in the unit.
(Read TEST CONTENT.)

5 Answer the sample test question at the end of each lesson. Check your answer against the answer given to see how well you did.
(See TEST CONTENT.)

OVERVIEW OF THE EOCT

Successful test takers understand the importance of knowing as much about a test as possible. This information can help you determine how to study and prepare for the EOCT and how to pace yourself during the test. The box below gives you a snapshot of the ***Coordinate Algebra* EOCT** and other important information.

THE EOCT AT A GLANCE

Administration Dates:

The EOCT has three primary annual testing dates: once in the spring, once in the summer, and once in the winter. There are also mid-month, online tests given in August, September, October, November, February, and March, as well as retest opportunities within the year.

Administration Time:

Each EOCT is composed of two sections, and students are given 60 minutes to complete each section. There is also a short stretch break between the two sections of the test.

Question Format:

All the questions on the EOCT are multiple-choice.

Number of Questions:

Each section of the ***Coordinate Algebra* EOCT** contains 31 questions; there are a total of 62 questions on the ***Coordinate Algebra* EOCT**.

Impact on Course Grade:

For students in grade 10 or above beginning the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 85% and the EOCT score 15%. For students in grade 9 beginning the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 80% and the EOCT score 20%. A student must have a final grade of at least 70 to pass the course and to earn credit toward graduation.

If you have additional administrative questions regarding the EOCT, please visit the Georgia Department of Education Web site at www.doe.k12.ga.us, see your teacher, or see your school test coordinator.

PREPARING FOR THE EOCT

WARNING!

You cannot prepare for this kind of test in one night. Questions will ask you to apply your knowledge, not list specific facts. Preparing for the EOCT will take time, effort, and practice.

To do your best on the ***Coordinate Algebra* EOCT**, it is important that you take the time necessary to prepare for this test and develop those skills that will help you take the EOCT.

First, you need to make the most of your classroom experiences and test preparation time by using good **study skills**. Second, it is helpful to know general **test-taking strategies** to ensure that you will achieve your best score.

Study Skills

A LOOK AT YOUR STUDY SKILLS

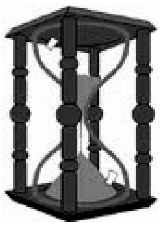
Before you begin preparing for this test, you might want to consider your answers to the following questions. You may write your answers here or on a separate piece of paper.

1. How would you describe yourself as a student?
   Response: ____
2. What are your study skills strengths and/or weaknesses as a student?
   Response: ____
3. How do you typically prepare for a mathematics test?
   Response: ____
4. Are there study methods you find particularly helpful? If so, what are they?
   Response: ____
5. Describe an ideal study situation (environment).
   Response: ____
6. Describe your actual study environment.
   Response: ____
7. What can you change about the way you study to make your study time more productive?
   Response: ____

Effective study skills for preparing for the EOCT can be divided into three categories:

- **Time Management**
- **Organization**
- **Active Participation**

Time Management



Do you have a plan for preparing for the EOCT? Often students have good intentions for studying and preparing for a test, but without a plan, many students fall short of their goals. Here are some strategies to consider when developing your study plan:

- Set realistic goals for what you want to accomplish during each study session and chart your progress.
- Study during your most productive time of the day.
- Study for reasonable amounts of time. Marathon studying is not productive.
- Take frequent breaks. Breaks can help you stay focused. Doing some quick exercises (e.g., sit-ups or jumping jacks) can help you stay alert.
- Be consistent. Establish your routine and stick to it.
- Study the most challenging test content first.
- For each study session, build in time to review what you learned in your last study session.
- Evaluate your accomplishments at the end of each study session.
- Reward yourself for a job well done.

Organization

You don’t want to waste your study time. Searching for materials, trying to find a place to study, and debating what and how to study can all keep you from having a productive study session. Get organized and be prepared. Here are a few organizational strategies to consider:

- Establish a study area that has minimal distractions.
- Gather your materials in advance.
- Develop and implement your study plan (see Appendices A–D for sample study plan sheets).

Active Participation

Students who actively study will learn and retain information longer. Active studying also helps you stay more alert and be more productive while learning new information. What is active studying? It can be anything that gets you to interact with the material you are studying. Here are a few suggestions:

- Carefully read the information and then DO something with it. Mark the important points with a highlighter, circle it with a pen, write notes on it, or summarize the information in your own words.
- Ask questions. As you study, questions often come into your mind. Write them down and actively seek the answers.
- Create sample test questions and answer them.
- Find a friend who is also planning to take the test and quiz each other.

Test-taking Strategies

There are many test-taking strategies that you can use before and during a test to help you have the most successful testing situation possible. Below are a few questions to help you take a look at your test-taking skills.

A LOOK AT YOUR TEST-TAKING SKILLS

As you prepare to take the EOCT, you might want to consider your answers to the following questions. You may write your answers here or on your own paper.

1. How would you describe your test-taking skills?
   Response: ____

2. How do you feel when you are taking a test?
   Response: ____

3. List the strategies that you already know and use when you are taking a test.
   Response: ____

4. List test-taking behaviors you use that contribute to your success when preparing for and taking a test.
   Response: ____

5. What would you like to learn about taking tests?
   Response: ____

Suggested Strategies to Prepare for the EOCT

Learn from the past. Think about your daily/weekly grades in your mathematics classes (past and present) to answer the following questions:

- In which specific areas of mathematics were you or are you successful?

  Response: ____

- Is there anything that has kept you from achieving higher scores?

  Response: ____

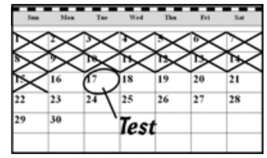
- What changes should you implement to achieve higher scores?

  Response: ____

Before taking the EOCT, work toward removing or minimizing any obstacles that might stand in the way of performing your best. The test preparation ideas and test-taking strategies in this section are designed to help guide you to accomplish this.

Be prepared. The best way to perform well on the EOCT is to be prepared. In order to do this, it is important that you know what standards/skills will be measured on the ***Coordinate Algebra*** **EOCT** and then practice understanding and using those standards/skills. The TEST CONTENT section of this study guide is designed to help you understand the specific standards that are on the ***Coordinate Algebra*** **EOCT** and give you suggestions for how to study the standards that will be assessed. Take the time to read through this material and follow the study suggestions. You can also ask your math teacher for any suggestions he or she might offer on preparing for the EOCT.

Start now. Don't wait until the last minute to start preparing. Begin early and pace yourself. By preparing a little bit each day, you will retain the information longer and increase your confidence level. Find out when the EOCT will be administered, so you can allocate your time appropriately.



Suggested Strategies the Day before the EOCT

✓ **Review what you learned from this study guide**.

1. Review the general test-taking strategies discussed in the "Top 10 Suggested Strategies during the EOCT" on page 14.
2. Review the content information discussed in the TEST CONTENT section beginning on page 15.
3. Focus your attention on the main topic, or topics, that you are most in need of improving.

✓ **Take care of yourself**.

1. Try to get a good night's sleep. Most people need an average of eight hours, but everyone's sleep needs are different.
2. Don't drastically alter your routine. If you go to bed too early, you might lie in bed thinking about the test. You want to get enough sleep so you can do your best.

Suggested Strategies the Morning of the EOCT

Eat a good breakfast. Choose foods high in protein for breakfast (and for lunch if the test is given in the afternoon). Some examples of foods high in protein are peanut butter, meat, and eggs. Protein gives you long-lasting, consistent energy that will stay with you through the test to help you concentrate better. Avoid foods high in sugar content. It is a misconception that sugar sustains energy—after an initial boost, sugar will quickly make you more tired and drained. Also, don't eat too much. A heavy meal can make you feel tired. So think about what you eat before the test.

Dress appropriately. If you are too hot or too cold during the test, it can affect your performance. It is a good idea to dress in layers, so you can stay comfortable, regardless of the room temperature, and keep your mind on the EOCT.

Arrive for the test on time. Racing late into the testing room can cause you to start the test feeling anxious. You want to be on time and prepared.

TOP 10

Suggested Strategies during the EOCT

These general test-taking strategies can help you do your best during the EOCT.

1 **Focus on the test.** Try to block out whatever is going on around you. Take your time and think about what you are asked to do. Listen carefully to all the directions.

2 **Budget your time.** Be sure that you allocate an appropriate amount of time to work on each question on the test.

3 **Take a quick break if you begin to feel tired.** To do this, put your pencil down, relax in your chair, and take a few deep breaths. Then, sit up straight, pick up your pencil, and begin to concentrate on the test again. Remember that each test section is only 60 minutes.

4 **Use positive self-talk.** If you find yourself saying negative things to yourself such as "I can't pass this test," it is important to recognize that you are doing this. Stop and think positive thoughts such as "I prepared for this test, and I am going to do my best." Letting the negative thoughts take over can affect how you take the test and your test score.

5 **Mark in your test booklet.** Mark key ideas or things you want to come back to in your test booklet. Remember that only the answers marked on your answer sheet will be scored.

6 **Read the entire question and the possible answer choices.** It is important to read the entire question so you know what it is asking. Read each possible answer choice. Do not mark the first one that "looks good."

7 **Use what you know.** Draw on what you have learned in class, from this study guide, and during your study sessions to help you answer the questions.

8 **Use content domain-specific strategies to answer the questions.** In the TEST CONTENT section, there are a number of specific strategies that you can use to help improve your test performance. Spend time learning these helpful strategies, so you can use them while taking the test.

9 **Think logically.** If you have tried your best to answer a question but you just aren't sure, use the process of elimination. Look at each possible answer choice. If it doesn't seem like a logical response, eliminate it. Do this until you've narrowed down your choices. If this doesn't work, take your best educated guess. It is better to mark something down than to leave it blank.

10 **Check your answers.** ☑ When you have finished the test, go back and check your work.

A WORD ON TEST ANXIETY

It is normal to have some stress when preparing for and taking a test. It is what helps motivate us to study and try our best. Some students, however, experience anxiety that goes beyond normal test "jitters." If you feel you are suffering from test anxiety that is keeping you from performing at your best, please speak to your school counselor, who can direct you to resources to help you address this problem.

TEST CONTENT

Up to this point in this study guide, you have been learning various strategies on how to prepare for and take the EOCT. This section focuses on what will be tested. It also includes sample questions that will let you apply what you have learned in your classes and from this study guide.

This section of the study guide will help you learn and review the various mathematical concepts that will appear on the ***Coordinate Algebra*** **EOCT**. Since *mathematics* is a broad term that covers many different topics, the state of Georgia has divided it into five major **conceptual categories** that portray a coherent view of high school mathematics. Each of the conceptual categories is broken down into big ideas. These big ideas are called **content standards**. Each conceptual category contains standards that cover different ideas related to that category. Each question on the EOCT measures an individual standard within a conceptual category.

The five conceptual categories for the ***Coordinate Algebra*** **EOCT** are the following:

- Algebra
- Functions
- Number and Quantity
- Geometry
- Statistics and Probability

These categories are important for several reasons. Together, they cover the major skills and concepts needed to understand and solve mathematical problems. These skills have many practical applications in the real world. Another more immediate reason that they are important has to do with test preparation. The best way to prepare for any test is to study and know the material measured on the test.

This study guide is organized in six **units** that review the material covered within the six units of the Coordinate Algebra course map. It is presented by topic rather than by category or standard (although those are listed at the beginning of each unit and are integral to each topic). The more you understand about the topics in each unit, the greater your chances of getting a good score on the EOCT.

Studying the Content Standards and Topics (Units 1 through 6)

You should be familiar with many of the content standards and topics that follow. It makes sense to spend more time studying the content standards and topics that you think may cause you problems. Even so, do not skip over any of them. The TEST CONTENT section has been organized into six units. Each unit is organized by the following features:

- **Introduction:** an overview of what will be discussed in the unit
- **Key Standards:** information about the specific standards that will be addressed. Standards that highlight mathematical modeling appear throughout the course and are highlighted with a (★) symbol. Strikethroughs in the standards are to highlight the portions that are not relevant to this course.
- **Main Topics:** the broad subjects covered in the unit

  Each Main Topic includes:
  - **Key Ideas:** definitions of important words and ideas as well as descriptions, examples, and steps for solving problems.
  - **Review Examples:** problems with solutions showing possible ways to answer given questions
  - **EOCT Practice Items:** sample multiple-choice questions similar to test items on the ***Coordinate Algebra*** **EOCT** with answer keys provided

With some time, determination, and guided preparation, you will be better prepared to take the ***Coordinate Algebra*** **EOCT**.

Unit 1: Relationships Between Quantities

In this unit, you will study quantitative relationships. You will learn how important units are for interpreting problems and setting up equations. The focus will be on both one- and two-variable linear and exponential equations. There will also be examples of modeling with inequalities.

KEY STANDARDS

Reason quantitatively and use units to solve problems.

MCC9-12.N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.★

MCC9-12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.★

MCC9-12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.★

Interpret the structure of expressions.

MCC9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.★ (*Emphasis on linear expressions and exponential expressions with integer exponents*.)

MCC9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients.★ (*Emphasis on linear expressions and exponential expressions with integer exponents*.)

MCC9-12.A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.★ (*Emphasis on linear expressions and exponential expressions with integer exponents*.)

Create equations that describe numbers or relationships.

MCC9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear ~~and quadratic functions, and simple rational~~ and exponential functions.★

MCC9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★ (*Limit to linear and exponential equations and, in the case of exponential equations, limit to situations requiring evaluation of exponential functions at integer inputs*.)

MCC9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*. ★ (*Limit to linear equations and inequalities*.)

MCC9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.* ★ (*Limit to formulas with a linear focus*.)

QUANTITIES AND UNITS

KEY IDEAS

1. A ***quantity*** is an exact amount or measurement. One type of quantity is a simple count, such as 5 eggs or 12 months. A second type of quantity is a measurement, which is an amount of a specific unit. Examples are 6 feet and 3 pounds.

2. A quantity can be exact or approximate. When an approximate quantity is used, it is important that we consider its level of accuracy. When working with measurements, we need to determine what level of accuracy is necessary and practical. For example, a dosage of medicine would need to be very precise. An example of a measurement that does not need to be very precise is the distance from your house to a local mall. The use of an appropriate unit for measurements is also important. For example, if you want to calculate the diameter of the Sun, you would want to choose a very large unit as your measure of length, such as miles or kilometers. Conversion of units can require approximations.

   Example:

   Convert 5 miles to feet.

   Solution:

   We know 1 mile is 5,280 feet.

$$5 \text{ miles} \times \frac{5,280 \text{ feet}}{1 \text{ mile}} = 26,400 \text{ feet}$$

   We can approximate that 5 miles is close to 26,000 feet.

3. The context of a problem tells us what types of units are involved. ***Dimensional analysis*** is a way to determine relationships among quantities using their dimensions, units, or unit equivalencies. Dimensional analysis suggests which quantities should be used for computation in order to obtain the desired result.

Example:

The number of calories a person burns doing an activity can be approximated using the formula $C = kmt$, where m is the person's weight in pounds and t is the duration of the activity in minutes. Find the units for the coefficient k.

Solution:

The coefficient k is a rate of burning calories specific to the activity. To find the units for k, solve the equation $C = kmt$ for k, and then look at the units.

$$C \text{ calories} = k \times m \text{ pounds} \times t \text{ minutes}$$

$$C = kmt$$

$$\frac{C}{mt} = k$$

$$\frac{C \text{ calories}}{mt \text{ pounds} \bullet \text{minutes}} = k$$

The value of k is $\frac{C}{mt}$, and the unit is calories per pound-minute.

You can check this using dimensional analysis:

$$C = kmt$$

$$C = \frac{k \text{ calories}}{\cancel{\text{pounds}} \cdot \cancel{\text{minutes}}} \cdot m \ \cancel{\text{pounds}} \cdot t \ \cancel{\text{minutes}}$$

$$C = kmt \text{ calories}$$

4. The process of dimensional analysis is also used to convert from one unit to another. Knowing the relationship between units is essential for unit conversion.

Example:

Convert 45 miles per hour to feet per minute.

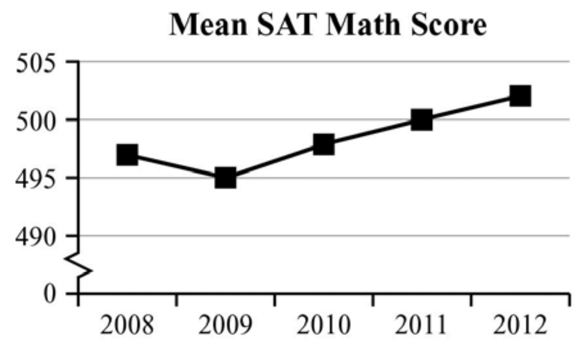
Solution:

To convert the given units, we use a form of dimensional analysis. We will multiply 45 mph by a series of ratios where the numerator and denominator are in different units but equivalent to each other. The ratios are carefully chosen to introduce the desired units.

$$\frac{45 \text{ miles}}{\text{hr}} \times \frac{1 \text{ hr}}{60 \text{ minutes}} \times \frac{5,280 \text{ feet}}{1 \text{ mile}} = \frac{45 \times 5,280 \text{ feet}}{60 \times 1 \text{ minute}} = 3,960 \text{ feet per minute}$$

5. When data are displayed in a graph, the units and scale features are keys to interpreting the data. Breaks or an abbreviated scale in a graph should be noted as they can cause a misinterpretation of the data.

This sample graph shows the mean mathematics SAT scores for a school over a five-year period. All of the scores are between 200 and 800. When the vertical scale does not start at 200, it can exaggerate the change in the mean scores from year to year. The chart makes the changes in the mean mathematics SAT scores over the past five years look significant. In reality, the changes were very modest. They ranged only between 495 and 502 over those years.



6. The measurements we use are often approximations. It is routinely necessary to determine reasonable approximations.

Example:

When Justin goes to work, he drives at an average speed of 65 miles per hour. It takes about 1 hour and 30 minutes for Justin to arrive at work. His car travels about 25 miles per gallon of gas. If gas costs \$3.65 per gallon, how much money does Justin spend on gas to travel to work?

Solution:

First, calculate the distance Justin travels.

$$65 \text{ miles per hour} \bullet 1.5 \text{ hour} = 97.5 \text{ miles}$$

Justin can travel 25 miles on 1 gallon of gas. Because 97.5 miles is close to 100 miles, he needs about $100 \div 25 = 4$ gallons of gas.

To find the cost of gas to travel to work, multiply cost per gallon by the number of gallons.

$$4 \times \$3.65 = \$14.60$$

Important Tips

- When referring to a quantity, include the unit or the items being counted whenever possible.
- It is important to use appropriate units for measurements, and to understand the relative sizes of units for the same measurement. You will need to know how to convert between units, and how to round or limit the number of digits you use.
- Use units to help determine if your answer is reasonable. For example, if a question asks for a weight, and you find an answer in feet, check your answer.

REVIEW EXAMPLES

1) The formula for density d is $d = \frac{m}{v}$, where m is mass and v is volume. If mass is measured in kilograms and volume is measured in cubic meters, what is the unit for density?

Solution:

The unit for density is $\frac{\text{kilograms}}{\text{meters}^3}$, or $\frac{\text{kg}}{\text{m}^3}$.

2) A rectangle has a length of 2 meters and a width of 40 centimeters. What is the perimeter of the rectangle?

Solution:

The perimeter of a rectangular is found by using the formula $P = 2l + 2w$, where P is perimeter, l is length, and w is width.

To find the perimeter, both measurements need to have the same units. Convert 2 meters to centimeters.

$$2\,\text{m} \times \frac{100\,\text{cm}}{1\,\text{m}}$$

Cancel the like units and multiply the remaining factors. The product is the converted measurement.

$$2\,\text{m} \times \frac{100\,\text{cm}}{1\,\text{m}} = 200\text{ cm}$$

Use the converted measurement in the formula to find the perimeter.

$$P = 2l + 2w$$
$$P = 2(200) + 2(40)$$
$$P = 400 + 80$$
$$P = 480 \text{ cm}$$

EOCT Practice Items

1) A rectangle has a length of 12 m and a width of 400 cm. What is the perimeter of the rectangle?

A. 824 cm
B. 1600 cm
C. 2000 cm
D. 3200 cm

[Key: D]

2) Jill swam 200 meters in 2 minutes 42 seconds. If each lap is 50 meters long, which is MOST likely to be her time per lap?

A. 32 seconds

B. 40 seconds

C. 48 seconds

D. 60 seconds

[Key: B]

STRUCTURE OF EXPRESSIONS

KEY IDEAS

1. ***Arithmetic expressions*** contain numbers and operation signs.

   Examples: $2 + 4$ and $4 - (10 + 3)$

   Algebraic expressions contain one or more variables.

   Examples: $2x + 4$, $4x - (10 + 3y)$, and $\dfrac{9 + 2t}{5}$

   The parts of expressions that are separated by addition or subtraction signs are called ***terms***. Terms are usually composed of numerical factors and variable factors. The numerical factor is called the ***coefficient***.

   Consider the algebraic expression $4x + 7y - 3$. It has three terms: $4x$, $7y$, and 3. For $4x$, the coefficient is 4 and the variable factor is x. For $7y$, the coefficient is 7 and the variable is y. The third term, 3, has no variables and is called a ***constant***.

2. To interpret a formula, it is important to know what each variable represents and to understand the relationships between the variables.

   For example, look at the compound interest formula $A = P(1 + r)^t$, where A is the balance (the amount in the account), P is the principal (or starting amount deposited), r is the interest rate, and t is the number of years.

   This formula is used to calculate the balance of an account when the annual interest is compounded. The balance is the product of P and the power $(1 + r)^t$. Because the value r is positive, the formula is a growth formula.

   Another example is the formula used to estimate the number of calories burned while jogging, $C = 0.075mt$, where m represents a person's body weight in pounds and t is the number of minutes spent jogging. This formula tells us that the number of calories burned depends on a person's body weight and how much time is spent jogging. The coefficient, 0.075, is the factor used for jogging.

Important Tip

- To consider how a coefficient affects a term, try different coefficient values for the same term and explore the effects.

REVIEW EXAMPLES

1) An amount of $1,000 is deposited into a bank account that pays 4% annual interest. If there are no other withdrawals or deposits, what will be the balance of the account after 3 years?

Solution:

Use the formula $A = P(1 + r)^t$. P is $1,000, r is 4% or 0.04, and t is 3 years.

$$A = 1{,}000(1 + 0.04)^3 = 1{,}000 \times 1.124864 \approx 1{,}124.86$$

The balance after 3 years will be $1,124.86.

2) The number of calories burned during exercise depends on the activity. The formulas for two activities are given.

$$C_1 = 0.012mt \text{ and } C_2 = 0.032mt$$

a. If one activity is cooking and the other is bicycling, identify the formula that represents each activity. Explain your answer.

b. What value would you expect the coefficient to have if the activity were reading? Include units and explain your answer.

Solution:

a. The coefficient of the variable term mt tells us how strenuous the activity is. Since bicycling is more strenuous than cooking, its formula would have a higher coefficient. Therefore, the formula for bicycling is likely $C_2 = 0.032mt$.

b. Since reading is less strenuous than cooking, the number of calories burned is probably fewer, and the coefficient is probably smaller. Expect a coefficient smaller than 0.012 calories per pound-minute for reading.

EOCT Practice Items

1) The distance a car travels can be found using the formula $d = rt$, where d is the distance, r is the rate of speed, and t is time. How many miles does the car travel, if it drives at a speed of 70 miles per hour for $\frac{1}{2}$ hour?

A. 35 miles
B. 70 miles
C. 105 miles
D. 140 miles

[Key: A]

2) A certain population of bacteria has an average growth rate of 0.02 bacteria per hour. The formula for the growth of the bacteria's population is $A = P_0(2.71828)^{0.02t}$, where P_0 is the original population, and t is the time in hours.

If you begin with 200 bacteria, about how many bacteria will there be after 100 hours?

A. 7
B. 272
C. 1,478
D. 20,000

[Key: C]

EQUATIONS AND INEQUALITIES

KEY IDEAS

1. Problems in quantitative relationships generally call for us to determine how the quantities are related to each other, use a variable or variables to represent unknowns, and write and solve an equation. Some problems can be modeled using an equation with one unknown.

   Example:

   Two angles of a triangle measure 30° and 70°. What is the measure of the third angle?

   Solution:

   The sum of the angle measures in a triangle are 180°. Let $x°$ represent the measure of the third angle.

   $30° + 70° + x° = 180°$

   Next, we solve for x.

$$\begin{aligned} 30 + 70 + x &= 180 && \text{Write the original equation.} \\ 100 + x &= 180 && \text{Combine like terms.} \\ x &= 80 && \text{Subtract 100 from both sides.} \end{aligned}$$

   The third angle measures 80°.

2. There are also problems that can be modeled with inequalities. These can be cases where you want to find the minimum or maximum amount of something, and use key words like “at least,” “greater than,” “less than,” and “no more than.”

   Example:

   A social media website currently has 1,000 members. The number of people that join the website triples every month. After how many months will the website have more than 1,000,000 members?

   Solution:

   To triple means to multiply by 3. To find the number of months, x, it will take for the number of members to triple, find the number of factors of 3 that should be multiplied by 1,000 so that the product is more than 1,000,000. This can be written as an inequality:

$$1{,}000 \cdot 3^x > 1{,}000{,}000$$

To solve, first divide both sides by 1,000 to get $3^x > 1{,}000$.

Next, determine the number of times to multiply 3 by itself so that the product is greater than 1,000. One way to do this is to use a chart.

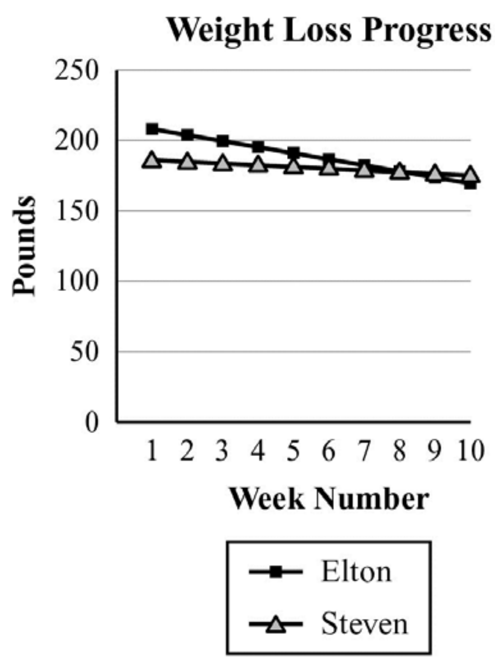
3^1	3^2	3^3	3^4	3^5	3^6	3^7
3	9	27	81	243	729	2,187

Because $3^7 > 1{,}000$, it will take 7 months for there to be more than 1,000,000 members.

3. Some situations involve a pair of related variables. To solve for two variables, you can use a system of two equations. You can solve some of these systems using graphs.

Example:

Elton loses 5 pounds each week. He started at 218 pounds on Week 1. Steven was 186 pounds on Week 1 and he loses 1 pound each week. The graph shows Elton's and Steven's weights by week.



a. What equations can be used to represent Elton's and Steven's weight loss?

b. After how many weeks do both Elton and Steven weigh the same number of pounds?

Solution:

a. Let p represent weight in pounds, and let t represent the number of weeks. Elton's weight can be represented by the equation $p = 218 - 5t$. Steven's weight can be represented by the equation $p = 186 - t$.

b. The graph seems to show that the lines intersect at week 8. To make sure this is correct, find the value of each equation for $t = 8$.
Elton: $218 - 5 \times 8 = 178$ pounds
Steven: $186 - 1 \times 8 = 178$ pounds
After 8 weeks, Elton and Steven both weigh the same, 178 pounds.

4. There are situations with two related variables that can be modeled with inequalities. These inequalities are called ***constraints*** and can be graphed in a coordinate plane. The intersections of the boundaries of the graphs of the constraints form the region where the solutions to the problem lie.

Example:

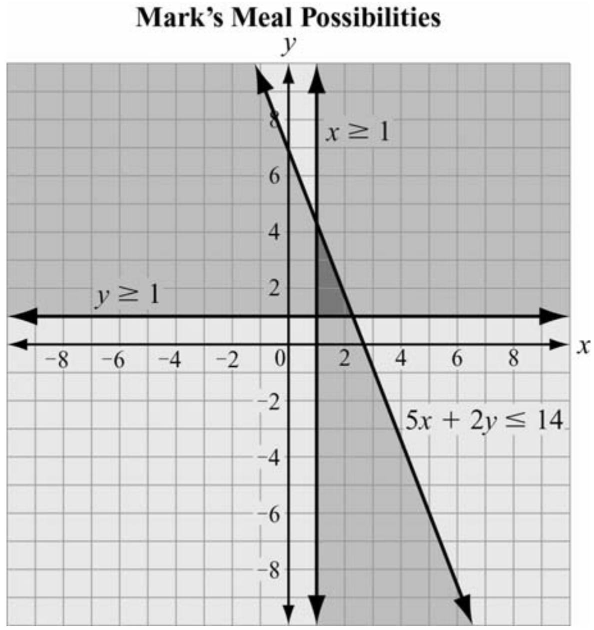
Mark has $14 to buy lunch for himself and his sister. He wants to buy at least one sandwich and one drink. If sandwiches cost $5 and drinks cost $2, what combinations of numbers of sandwiches and drinks could Mark buy?

Solution:

Use a system of inequalities to represent the situation. Let x be the number of sandwiches and y be the number of drinks. This system shows that the total cost must be less than or equal to $14, and there is at least 1 sandwich and 1 drink.

$$\begin{Bmatrix} 5x + 2y \le 14 \\ x \ge 1 \\ y \ge 1 \end{Bmatrix}$$

When you graph the system of inequalities, the shading is below the boundary for $5x + 2y \le 14$, to the right of $x \ge 1$, and above $y \ge 1$. The possible combinations of sandwiches and drinks are represented by the coordinates of the points where all three regions overlap.



The points in the triangular region with whole-number coordinates are: (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), and (2, 2). The first number in each ordered pair tells the number of sandwiches and the second number tells the number of drinks.

5. In some cases a number is not required for a solution. Instead, we want to know how a certain variable relates to another.

Example:

Solve the equation $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ for y_2.

Solution:

To solve for y_2, use inverse operations. Make sure you perform the same operation on both sides of the equation.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	Write the original equation.
$m(x_2 - x_1) = y_2 - y_1$	Multiply both sides by $x_2 - x_1$.
$m(x_2 - x_1) + y_1 = y_2$	Add y_1 to both sides.

REVIEW EXAMPLES

1) The city of Arachna has a spider population that has been doubling every year. If there are about 100,000 spiders this year, how many will there be 4 years from now?

Solution:

Doubling means multiplying by 2, so use this equation:

$$S = 100{,}000 \times 2 \times 2 \times 2 \times 2 \text{ or } 100{,}000 \times 2^4$$

$$S = 1{,}600{,}000$$

There will be 1,600,000 spiders 4 years from now.

2) The Jones family has twice as many tomato plants as pepper plants. If there are 21 plants in their garden, how many plants are pepper plants?

Solution:

Let p be the number of pepper plants, and let t be the number of tomato plants. Model the situation with these two equations:

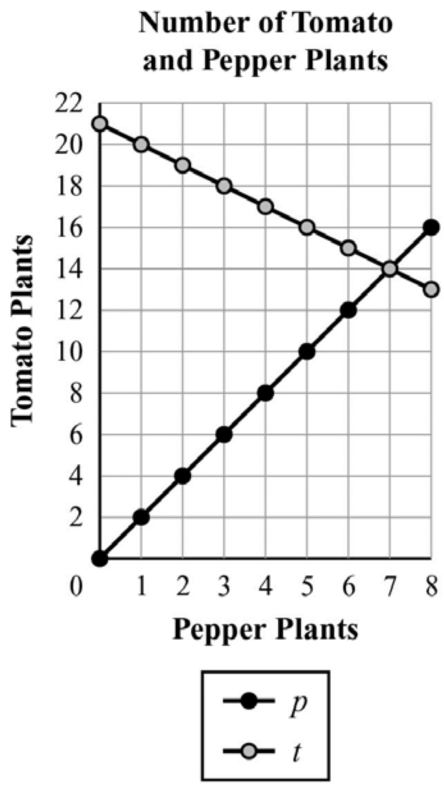
$$t = 2p$$

$$p + t = 21$$

Now create tables of values for each equation, and then graph the values. Be sure to use the same values of p for both tables. (Note: you can solve $p + t = 21$ for t to make it easier to find the values.)

$t = 2p$	
p	***t***
0	0
1	2
2	4
3	6
4	8
5	10
6	12
7	14
8	16

$t = 21 - p$	
p	***t***
0	21
1	20
2	19
3	18
4	17
5	16
6	15
7	14
8	13



From the graph, the equations share a common point (7, 14). That means there are 7 pepper plants and 14 tomato plants.

3) The sum of the angle measures in a triangle is 180°. The angles of a certain triangle measure $x°$, $2x°$, and $6x°$. Solve for x.

Solution:

Use the equation $x + 2x + 6x = 180$ to represent the relationship.

Combine like terms: $9x = 180$.

Divide each side by 9: $\frac{9x}{9} = \frac{180}{9}$; $x = 20$.

4) A business invests \$6,000 in equipment to produce a product. Each unit of the product costs \$0.90 to produce and is sold for \$1.50. How many units of the product must be sold in order for the business to make a profit?

Solution:

Let C be the total cost of producing x units. Represent the total cost with an equation.

$$C = 0.90x + 6000$$

Let R be the total revenue from selling x units. Represent the total revenue with an equation.

$$R = 1.50x$$

A break-even point is reached when the total revenue R equals the total cost C. A profit occurs when revenue is greater than cost.

$$1.50x > 0.90x + 6000$$

Solve the inequality.

Subtract $0.90x$ from each side: $0.60x > 6000$

Divide each side by 0.60: $x > 10{,}000$

More than 10,000 units must be sold in order for the business to make a profit.

EOCT Practice Items

1) The sum of the angle measures in a triangle is 180°. Two angles of a triangle measure 20° and 50°. What is the measure of the third angle?

A. 30°

B. 70°

C. 110°

D. 160°

[Key: C]

2) Which equation shows $P = 2l + 2w$ when solved for w?

A. $w = \dfrac{2l}{P}$

B. $w = \dfrac{2l - P}{2}$

C. $w = 2l - \dfrac{P}{2}$

D. $w = \dfrac{P - 2l}{2}$

[Key: D]

3) Bruce owns a business that produces widgets. He must bring in more in revenue than he pays out in costs in order to turn a profit.

- **It costs $10 in labor and materials to make each of his widgets.**
- **His rent each month for his factory is $4000.**
- **He sells each widget for $25**.

How many widgets does Bruce need to sell each month to make a profit?

A. 160

B. 260

C. 267

D. 400

[Key: C]

Unit 2: Reasoning with Equations and Inequalities

Unit 2 focuses on equations and inequalities. From transforming equations or inequalities to graphing their solutions, this unit covers linear relationships with one or two variables. Familiarity with the properties of operations and equality is essential for mastering the skills and concepts covered in this unit. This unit builds on your knowledge of coordinates and extends it to the use of algebraic methods to solve systems of equations.

KEY STANDARDS

Understand solving equations as a process of reasoning and explain the reasoning

MCC9-12.A.REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. (*Students should focus on and master linear equations and be able to extend and apply their reasoning to other types of equations in future courses*.)

Solve equations and inequalities in one variable

MCC9-12.A.REI.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. (*Extend earlier work with solving linear equations to solving linear inequalities in one variable and to solving literal equations that are linear in the variable being solved for. Include simple exponential equations that rely only on application of the laws of exponents, such as* $5^x = 125$ *or* $2^x = 1/16$.)

Solve systems of equations

MCC9-12.A.REI.5 Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. (*Limit to linear systems*.)

MCC9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically

MCC9-12.A.REI.12 Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

TRANSFORMATIONS OF EQUATIONS AND INEQUALITIES

KEY IDEAS

1. ***Equivalent expressions*** produce the same result when substituting values for variables.

   Example:

   Is the expression $\frac{6x+8}{2}$ equivalent to $3x+4$?

   Solution:

   Yes, the expressions are equivalent.

$\frac{6x+8}{2}$	Write the original expression.
$\frac{2(3x+4)}{2}$	Factor or use the distributive property (reversed).
$3x+4$	Simplify.

2. To solve a problem, it is often necessary to manipulate an equation or inequality to show an equivalent form. This is often done by isolating the variable. Here is a list of what you may do to correctly transform an equation or inequality in order to preserve its solution.

   - Substitution: Replace a quantity, including entire expressions, with an equivalent quantity. (See Key Idea 1.)
   - Use the properties of equality or inequality.

Name of Property	Procedure	Valid for	
		Equations	Inequalities
Addition	Add the same positive or negative number to both sides.	✓	✓
Subtraction	Subtract the same positive or negative number from both sides.	✓	✓
Multiplication	Multiply both sides by the same positive or negative number (not 0).	✓	✓ For positive numbers only. Multiplication by a negative number reverses the inequality sign.
Division	Divide both sides by the same positive or negative number (not 0).	✓	✓ For positive numbers only. Division by a negative number reverses the inequality sign.

- Use the Reflexive and Transitive properties of Equality and Inequality.
- Use the Symmetric Property of Equality.

Here are examples of when you might need to transform an equation.

Example:

Solve the equation $2y + 4 = 3(2x - 6)$ for y. Show and justify your steps.

Solution:

$2y + 4 = 3(2x - 6)$	Write the original equation.
$2y + 4 = 6x - 18$	Use the Distributive Property to write an equivalent expression on the right side.
$2y + 4 - 4 = 6x - 18 - 4$	Subtract 4 from both sides.
$2y = 6x - 22$	Combine like terms on both sides.
$\frac{2y}{2} = \frac{6x}{2} - \frac{22}{2}$	Divide each term on both sides by 2.
$y = 3x - 11$	Simplify both sides.

Example:

Solve the equation $14 = ax + 6$ for x. Show and justify your steps.

Solution:

$14 = ax + 6$	Write the original equation.
$14 - 6 = ax + 6 - 6$	Subtract 6 from both sides.
$8 = ax$	Combine like terms on each side.
$\frac{8}{a} = \frac{ax}{a}$	Divide each side by a.
$\frac{8}{a} = x$	Simplify.

Example:

Solve the inequality $4 - y > 5$ for y. Show and justify your steps.

Solution:

$4 - y > 5$	Write the original inequality.
$4 - 4 - y > 5 - 4$	Subtract 4 from both sides.
$-y > 1$	Combine like terms on both sides.
$(-1)(-y) < (-1)(1)$	Multiply both sides by –1 and reverse the inequality symbol.
$y < -1$	Simplify.

Important Tips

- Know the properties of operations and the order of operations so you can readily simplify algebraic expressions and prove two expressions are equivalent.
- Be familiar with the properties of equality and inequality. In particular, be aware that when you multiply or divide both sides of an inequality, you must reverse the inequality sign to preserve the relationship.
- Multiply both sides of an equation or inequality by a common denominator as a first step to eliminate denominators.

REVIEW EXAMPLES

1) Are the algebraic expressions $4x - 2$ and $6x - 2(x - 1)$ equivalent?

Solution:

Simplify the expression $6x - 2(x - 1)$ and see if it is equivalent to the other expression.

$$\begin{aligned} 6x - 2(x - 1) &= 6x - 2x + 2 \\ &= 4x + 2 \end{aligned}$$

No, the expressions are not equivalent.

2) Solve this inequality for y: $6a - 2y > 4$

Solution:

$6a - 2y > 4$	Write the original inequality.
$6a - 6a - 2y > 4 - 6a$	Subtract $6a$ from both sides.
$-2y > 4 - 6a$	Combine like terms on the left side.
$\frac{-2y}{-2} < \frac{4 - 6a}{-2}$	Divide each side by –2 and reverse the inequality symbol.
$y < -2 + 3a$	Simplify both sides.

3) Solve the equation $\frac{m}{6} + \frac{m}{4} = 1$ for m.

Solution:

$\frac{m}{6} + \frac{m}{4} = 1$	Write the original equation.
$2m + 3m = 12$	Multiply both sides by 12.
$5m = 12$	Combine like terms on the left side.
$\frac{5m}{5} = \frac{12}{5}$	Divide each side by 5.
$m = \frac{12}{5}$	Simplify.

EOCT Practice Items

1) Which equation shows $ax - w = 3$ solved for w?

A. $w = ax - 3$
B. $w = ax + 3$
C. $w = 3 - ax$
D. $w = 3 + ax$

[Key: A]

2) Which equation is equivalent to $\frac{7x}{4} - \frac{3x}{8} = 11$?

A. $17x = 88$
B. $11x = 88$
C. $4x = 44$
D. $2x = 44$

[Key: B]

3) Which equation shows $4n = 2(t - 3)$ solved for t?

A. $t = \dfrac{4n-2}{3}$

B. $t = \dfrac{4n+2}{3}$

C. $t = 2n - 3$

D. $t = 2n + 3$

[Key: D]

4) Which equation shows $6(x + 4) = 2(y + 5)$ solved for y?

A. $y = x + 3$
B. $y = x + 5$
C. $y = 3x + 7$
D. $y = 3x + 17$

[Key: C]

SOLVING EQUATIONS AND INEQUALITIES

KEY IDEAS

1. To solve an equation or inequality means to find the quantity or quantities that make the equation or inequality true. The strategies for solving an equation or inequality depend on the number of variables and the exponents that are included.

2. Here is an algebraic method for solving a linear equation with one variable:

   Write equivalent expressions until the desired variable is isolated on one side. Be sure to check your answers.

   Example:

   Solve $2(3-a)=18$.

   Solution:

   Solve the equation two ways:

$$\begin{aligned} 2(3-a)&=18\\ 3-a&=9\\ -a&=6\\ a&=-6 \end{aligned} \qquad\qquad \begin{aligned} 2(3-a)&=18\\ 6-2a&=18\\ -2a&=12\\ a&=-6 \end{aligned}$$

3. Here is an algebraic method for solving a linear inequality with one variable:

   Write equivalent expressions until the desired variable is isolated on one side. If you multiply or divide by a negative number, make sure you reverse the inequality symbol.

   Example:

   Solve $2(5-x)>8$ for x.

   Solution:

   Solve the inequality using either of these two ways:

$$\begin{aligned} 2(5-x)&>8\\ 5-x&>4\\ -x&>-1\\ x&<1 \end{aligned} \qquad\qquad \begin{aligned} 2(5-x)&>8\\ 10-2x&>8\\ -2x&>-2\\ x&<1 \end{aligned}$$

Important Tips

- If you multiply or divide both sides of an inequality by a negative number, make sure you reverse the inequality sign.
- Be familiar with the properties of equality and inequality so you can transform equations or inequalities.
- Sometimes eliminating denominators by multiplying all terms by a common denominator makes it easier to solve an equation or inequality.

REVIEW EXAMPLES

1) Karla wants to save up for a prom dress. She figures she can save \$9 each week from the money she earns babysitting. If she plans to spend less than \$150 for the dress, how many weeks will it take her to save enough money to buy any dress in her price range?

Solution:

Let w represent the number of weeks. If she saves \$9 each week, Karla will save $9w$ dollars after w weeks. We need to determine the minimum number of weeks it will take her to save \$150. Use the inequality $9w \geq 150$ to solve the problem. We need to transform $9w \geq 150$ to isolate w. Divide both sides by 9 to get $w \geq 16\frac{2}{3}$ weeks. Because we do not know what day Karla gets paid each week, we need the answer to be a whole number. So, the answer has to be 17, the smallest whole number greater than $16\frac{2}{3}$. She will save \$144 after 16 weeks, and \$153 after 17 weeks.

2) Joachim wants to know if he can afford to add texting to his cell phone plan. He currently spends \$21.49 per month for his cell phone plan, and the most he can spend for his cell phone is \$30 per month. He could get unlimited texts added to his plan for an additional \$10 each month. Or, he could get a "pay-as-you-go" plan that charges a flat rate of \$0.15 per text message. He assumes that he will send an average of 5 text messages per day. Can Joachim afford to add a text message plan to his cell phone?

Solution:

Joachim cannot afford either plan.

At an additional \$10 per month for unlimited texting, Joachim's cell phone bill would be \$31.49 a month. If he chose the pay-as-you-go plan, each day he would expect to pay for 5 text messages. Let t stand for the number of text messages per month. Then, on the pay-as-you-go plan, Joachim could expect his cost to be represented by the expression:

$$21.49 + 0.15t$$

If he must keep his costs at \$30 or less, $21.49 + 0.15t \leq 30$.

To find the number of text messages he can afford, solve for t.

$21.49 - 21.49 + 0.15t \leq 30 - 21.49$	Subtract 21.49 from both sides.
$0.15t \leq 8.51$	Simplify both sides.
$t \leq 56.733\ldots$	Divide both sides by 0.15.

The transformed inequality tells us that Joachim would need to send less than 57 text messages per month to afford the pay-as-you-go plan. However, 5 text messages per day at a minimum of 28 days in a month is 140 text messages per month. So, Joachim cannot afford text messages for a full month, and neither plan fits his budget.

3) Two cars start at the same point and travel in opposite directions. The first car travels 15 miles per hour faster than the second car. In 4 hours, the cars are 300 miles apart. Use the formula below to determine the rate of the second car.

$$4(r + 15) + 4r = 300$$

What is the rate, r, of the second car?

Solution:

The second car is traveling 30 miles per hour.

$4(r + 15) + 4r = 300$	Write the original equation.
$4r + 60 + 4r = 300$	Multiply 4 by $r + 15$.
$8r + 60 = 300$	Combine like terms.
$8r = 240$	Subtract 60 from each side.
$r = 30$	Divide each side by 8.

EOCT Practice Items

1) This equation can be used to find *h*, the number of hours it takes Flo and Bryan to mow their lawn.

$$\frac{h}{3}+\frac{h}{6}=1$$

How many hours will it take them to mow their lawn?

A. 6
B. 3
C. 2
D. 1

[Key: C]

2) A ferry boat carries passengers back and forth between two communities on the Peachville River.

- **It takes 30 minutes longer for the ferry to make the trip upstream than downstream.**
- **The ferry's average speed in still water is 15 miles per hour.**
- **The river's current is usually 5 miles per hour.**

This equation can be used to determine how many miles apart the two communities are.

$$\frac{m}{15-5}=\frac{m}{15+5}+0.5$$

What is *m*, the distance between the two communities?

A. 0.5 miles
B. 5 miles
C. 10 miles
D. 15 miles

[Key: C]

3) For what values of x is the inequality $\frac{2}{3}+\frac{x}{3}>1$ true?

A. $x < 1$
B. $x > 1$
C. $x < 5$
D. $x > 5$

[Key: B]

SOLVING A SYSTEM OF TWO LINEAR EQUATIONS

KEY IDEAS

1. A system of linear equations consists of two or more linear equations that may or may not have a common solution. The solution of a system of two linear equations is the set of values for the variables that makes all the equations true. The solutions can be expressed as ordered pairs in coordinate notation (x, y) or as two equations, one for x and the other for y (x = … and y = …).

   Strategies:

   - ❖ Use graphs of the equations to visually estimate a common point. First, prepare a table of values for each equation in the system, using the same set of numbers for the first coordinate. Graph both equations and test the coordinates of the point where the lines appear to cross in both equations to see if the coordinates are a common solution.

   Example:

   Solve this system of equations.

   $$\begin{cases} y = 2x - 4 \\ x = y + 1 \end{cases}$$

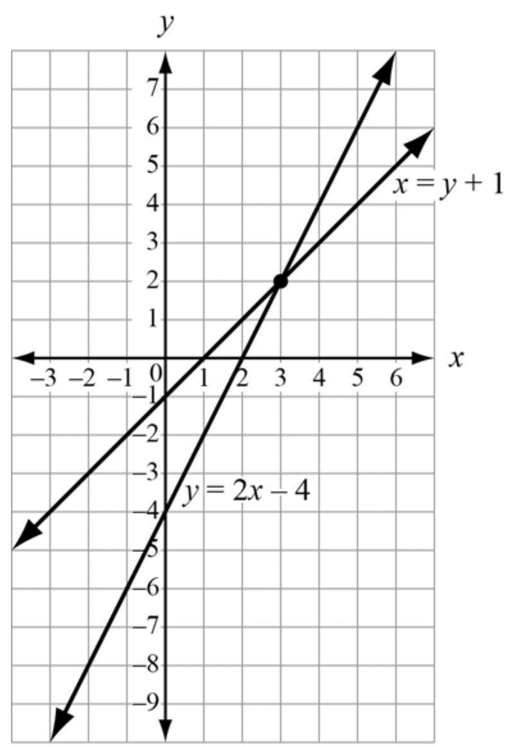
Solution:

First, find coordinates of points for each equation. Making a table of values for each is one way to do this. Use the same values for both equations.

x	**2x – 4**
–1	–6
0	–4
1	–2
2	0
3	2
4	4

x	**y**
–1	–2
0	–1
1	0
2	1
3	2
4	3

Using the tables, we display both equations on the graph below.



The graph shows all the ordered pairs of numbers (rows from the table) that satisfy $y = 2x - 4$ and the ordered pairs that satisfy $x = y + 1$. From the graph it appears that the lines cross at about (3, 2). Then try that combination in both equations and see that (3, 2) is a solution to both equations. So, (3, 2) is the solution to the system of equations. The graph also suggests that (3, 2) is the only point the lines have in common, so we have found the only pair of numbers that works for both equations.

Strategies:

- ❖ Simplify the problem by eliminating one of the two variables.

<u>Substitution method</u>: Use one equation to isolate a variable and replace that variable in the other equation with the equivalent expression you just found. Solve for the one remaining variable. Use the solution to the remaining variable to find the unknown you eliminated.

Example:

Solve this system of equations.

$$\begin{cases} 2x - y = 1 \\ 5 - 3x = 2y \end{cases}$$

Solution:

Begin by choosing one of the equations and solving for one of the variables. This variable is the one you will eliminate. We could solve the top equation for *y*.

$$\begin{aligned} 2x - y &= 1 \\ 2x &= 1 + y \\ 2x - 1 &= y \\ y &= 2x - 1 \end{aligned}$$

Next, use substitution to replace the variable you are eliminating in the other equation.

$$\begin{aligned} 5 - 3x &= 2y \\ 5 - 3x &= 2(2x - 1) \\ 5 - 3x &= 4x - 2 \\ 7 &= 7x \\ 1 &= x \end{aligned}$$

Now, find the corresponding *y*-value. You can use either equation.

$$\begin{aligned} 2x - y &= 1 \\ 2(1) - y &= 1 \\ 2 - y &= 1 \\ -y &= 1 - 2 \\ -y &= -1 \\ y &= 1 \end{aligned}$$

So, the solution is $x = 1$ and $y = 1$, or (1, 1).

Addition method: Add the equations (or a transformation of the equations) to eliminate a variable. Then solve for the remaining variable, and use this value to find the value of the variable you eliminated.

Example:

Solve this system of equations.

$$\begin{cases} 2x - y = 1 \\ 5 - 3x = -y \end{cases}$$

Solution:

First, rewrite the second equation in standard form.

$$\begin{cases} 2x - y = 1 \\ -3x + y = -5 \end{cases}$$

Decide which variable to eliminate. We can eliminate the y-terms because they are opposites.

$$\begin{aligned} 2x - y &= 1 \\ -3x + y &= -5 \end{aligned}$$

Add the equations, term by term, eliminating y and reducing to one equation. This is an application of the addition property of equality.

$$-x = -4$$

Multiply both sides by –1 to solve for x.

$$\begin{aligned} (-1)(-x) &= (-1)(-4) \\ x &= 4 \end{aligned}$$

Now substitute this value of x in either original equation to find y.

$$\begin{aligned} 2x - y &= 1 \\ 2(4) - y &= 1 \\ 8 - y &= 1 \\ -y &= -7 \\ y &= 7 \end{aligned}$$

The solution to the system of equations is (4, 7).

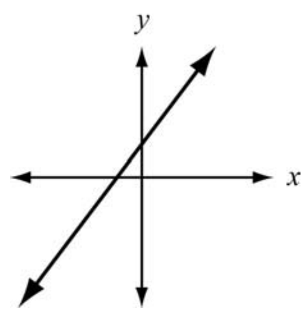
2. The graphing method only suggests the solution of a system of equations. To check the solution, substitute the values into the equations and make sure the ordered pair satisfies both equations.

3. When graphing a system of equations:

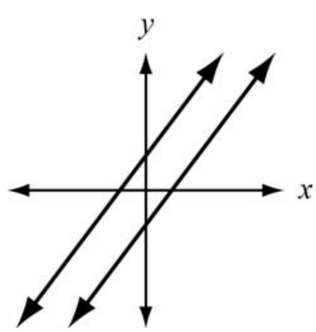
   a. If the lines are parallel, then there is no solution to the system.

   b. If the lines coincide, then the lines have all their points in common and any pair of points that satisfies one equation will satisfy the other.

4. When using elimination to solve a system of equations, if both variables are removed when you try to eliminate one, and if the result is a true equation such as $0 = 0$, then the lines coincide. The equations would have all ordered pairs in common.



When using elimination to solve a system of equations, if the result is a false equation such as $3 = 7$, then the lines are parallel. The system of equations has no solution, since there is no point where the lines intersect.



REVIEW EXAMPLES

1) Rebecca has five coins worth 65 cents in her pocket. If she only has quarters and nickels, how many quarters does she have? Use a system of equations to arrive at your answer and show all steps.

Solution:

If q represents the number of quarters and n represents the number of nickels, the two equations could be $25q + 5n = 65$ (value of quarters plus value of nickels is 65 cents) and $q + n = 5$ (she has 5 coins). The equations in the system would be $25q + 5n = 65$ and $q + n = 5$.

Next, solve $q + n = 5$ for q. By subtracting n from both sides, the result is $q = 5 - n$.

Next, eliminate q by replacing q with $5 - n$ in the other equation: $25(5 - n) + 5n = 65$
Solve this equation for n.

$$25(5 - n) + 5n = 65$$

$$125 - 25n + 5n = 65$$

$$125 - 20n = 65$$

$$-20n = -60$$

$$n = 3$$

Now solve for q by replacing n with 3 in the equation $q = 5 - n$. So, $q = 5 - 3 = 2$, and 2 is the number of quarters.

Rebecca has 2 quarters and 3 nickels.

2) Peg and Larry purchased "no contract" cell phones. Peg's phone cost \$25 plus \$0.25 per minute. Larry's phone cost \$35 plus \$0.20 per minute. After how many minutes of use will Peg's phone cost more than Larry's phone?

Solution:

Let x represent the number of minutes used. Peg's phone costs $25 + 0.25x$. Larry's phone costs $35 + 0.20x$. We want Peg's cost to exceed Larry's.
This gives us $25 + 0.25x > 35 + 0.20x$, which we then solve for x.

$$25 + 0.25x > 35 + 0.20x$$
$$25 + 0.25x - 0.20x > 35 + 0.20x - 0.20x$$
$$25 + 0.05x > 35$$
$$25 - 25 + 0.05x > 35 - 25$$
$$0.05x > 10$$
$$\frac{0.05x}{0.05} > \frac{10}{0.05}$$
$$x > 200$$

After 200 minutes of use, Peg's phone will cost more than Larry's phone.

3) Is (3, –1) a solution of this system?

$$\begin{cases} y = 2 - x \\ 3 - 2y = 2x \end{cases}$$

Solution:

Substitute the coordinates of (3, –1) into each equation.

$$\begin{aligned} y &= 2 - x \\ -1 &= 2 - 3 \\ -1 &= -1 \end{aligned} \qquad \begin{aligned} 3 - 2y &= 2x \\ 3 - 2(-1) &= 2(3) \\ 3 + 2 &= 6 \\ 5 &= 6 \end{aligned}$$

The coordinates of the given point do not satisfy $3 - 2y = 2x$. If you get a false equation when trying to solve a system algebraically, then it means that the coordinates of the point are not the solution. So, (3, –1) is not a solution of the system.

4) Solve this system.

$$\begin{cases} x - 3y = 6 \\ -x + 3y = -6 \end{cases}$$

Solution:

Add the terms of the equations. Each pair of terms is opposites, and the result is 0 + 0 = 0.

This result is always true, so the two equations represent the same line. Every point on the line is a solution to the system.

5) Solve this system.

$$\begin{cases} -3x - y = 10 \\ 3x + y = -8 \end{cases}$$

Solution:

Add the terms in the equations: 0 = 2

The result is never true. The two equations represent parallel lines. As a result, the system has no solution.

EOCT Practice Items

1) A manager is comparing the cost of buying ball caps with the company emblem from two different companies.

- **Company X charges a $50 fee plus $7 per cap.**
- **Company Y charges a $30 fee plus $9 per cap.**

For what number of ball caps will the manager's cost be the same for both companies?

A. 10 caps
B. 20 caps
C. 40 caps
D. 100 caps

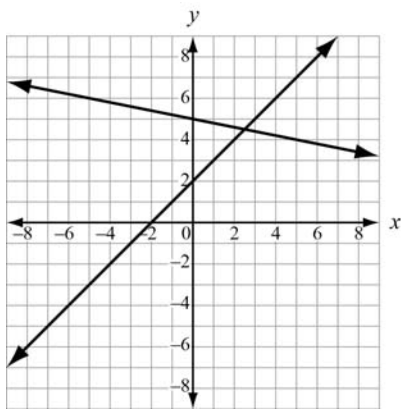
[Key: A]

2) A shop sells one-pound bags of peanuts for $2 and three-pound bags of peanuts for $5. If 9 bags are purchased for a total cost of $36, how many three-pound bags were purchased?

A. 3
B. 6
C. 9
D. 18

[Key: B]

3) Which graph represents a system of linear equations that has multiple common coordinate pairs?

A.


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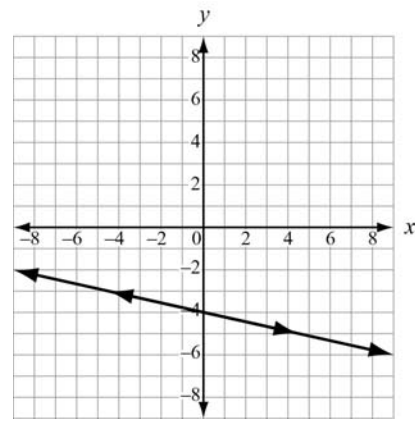

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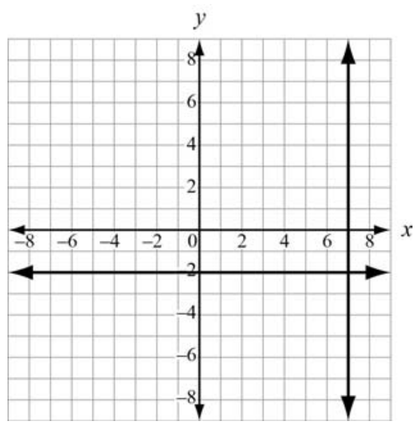

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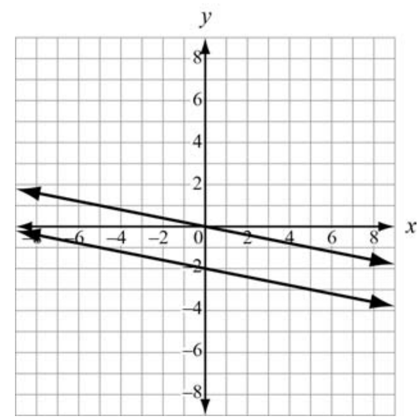


[Key: B]

GRAPHING THE SOLUTIONS OF EQUATIONS AND INEQUALITIES

KEY IDEAS

1. The solution to an equation or inequality can be displayed on a graph using a coordinate or coordinates. If the equation or inequality involves only <u>one</u> variable, then the number line is the coordinate system used.

Example:

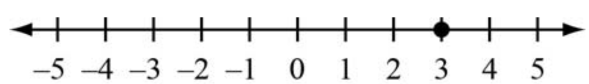
Use a number line to display the solution to $3x + 5 = 14$.

Solution:

First solve the equation:

$$
\begin{aligned}
3x + 5 &= 14 \\
3x + 5 - 5 &= 14 - 5 \\
3x &= 9 \\
x &= 3
\end{aligned}
$$

After transforming the equation we get $x = 3$. Since there is only one variable, use a number line to display the answer.



For an equation, the display shows the value(s) on the number line that satisfy the equation. Typically, a dot is placed on the line where the solution lies.

Example:

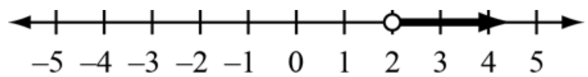
Use a number line to display the solution to $3x + 8 > 14$.

Solution:

First solve the inequality.

$$\begin{aligned} 3x + 8 &> 14 \\ 3x + 8 - 8 &> 14 - 8 \\ 3x &> 6 \\ x &> 2 \end{aligned}$$

After transforming the inequality, the result is $x > 2$. Again, use a number line to display the answer. For an inequality, the display shows the values on the number line that satisfy the inequality. The display is usually a ray drawn on the number line that may or may not include its starting point. The inequality uses the > symbol, so use an open circle to show that 2 is not a solution.



The solution is all values of x greater than 2.

Example:

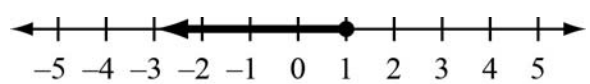
Use a number line to display the solution to $7 - 4x \geq 3$.

Solution:

First solve the inequality:

$$\begin{aligned} 7 - 4x &\geq 3 \\ 7 - 4x - 7 &\geq 3 - 7 \\ -4x &\geq -4 \\ x &\leq 1 \end{aligned}$$

The solution of the inequality is $x \leq 1$. Because the inequality uses the ≤ symbol, use a closed circle to show that 1 is a solution.



The solution is all values of x less than or equal to 1.

2. If the equation or inequality involves <u>two</u> variables, then a coordinate plane is used to display the solution. For an equation with two variables, the display shows the points (ordered pairs) that satisfy the equation. The display should look like a curve, line, or part of a line, depending on the situation.

Example:

Use a rectangular coordinate system to display the solution to $3x + y = 14$.

Solution:

First, solve the equation for y.

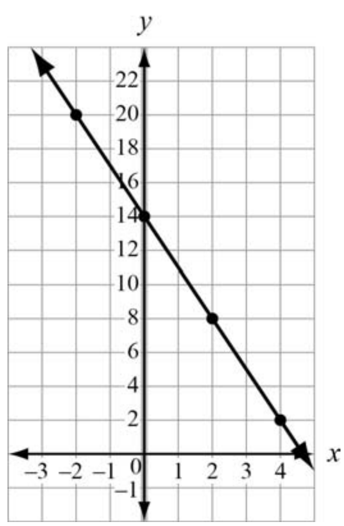
$$\begin{aligned} 3x + y &= 14 \\ 3x - 3x + y &= 14 - 3x \\ y &= 14 - 3x \end{aligned}$$

We will need to determine some ordered pairs of numbers for x and y that satisfy the equation. A good way to do this is to organize your findings in an input/output table with a column for x and a column for y, as shown below.

x	**14 – 3x**	y
–2	14 – 3(–2)	20
0	14 – 3(0)	14
2	14 – 3(2)	8
4	14 – 3(4)	2

Use the numbers in the first and last columns of the table as the coordinates of the points on your graph. Connect the points, unless the numbers you chose are the only ones you may use.

Here is the graph of $3x + y = 14$.



3. For an inequality, the graph of the solutions is shown by a shaded half-plane. Points that lie in the shaded area are solutions to the inequality. If values on the boundary line are not solutions (<, >), then the line is dashed. If the values on the boundary line are solutions (≤, ≥), then the line is solid.

Example:

Use a rectangular coordinate system to display the solution to $3x + y > -1$.

Solution:

First, solve the inequality for y.

$$3x + y > -1$$
$$3x - 3x + y > -1 - 3x$$
$$y > -1 - 3x \quad \text{or} \quad y > -3x - 1$$

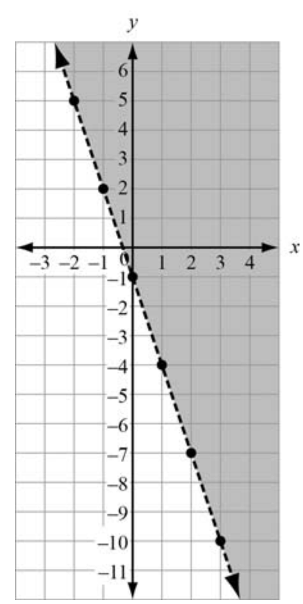
We will need to determine some ordered pairs of numbers for x and y that fit the boundary line $y = -3x - 1$.

x	$-3x - 1$	y
–1	–3(–1) – 1	2
0	–3(0) – 1	–1
1	–3(1) – 1	–4
2	–3(2) – 1	–7

Use the numbers in the first and last columns of the table as the coordinates of the points on your boundary line. Use a dashed line, since the inequality uses the > symbol.

Next, decide which side of the boundary line to shade. Choose a test point not on the line. If this point is a solution to the inequality, shade the region that includes this point. If the point is not a solution, shade the region that does not include this point. It is usually easy to use (0, 0) as a test point when it is not on the line.

Is $3(0) + 0 > -1$? Yes, so (0, 0) is a solution of the inequality. Shade the region above the line. The graph for $3x + y > -1$ is represented below.



Example:

Graph the solutions of $y + 2 \leq x$.

Solution:

First, solve the inequality for y.

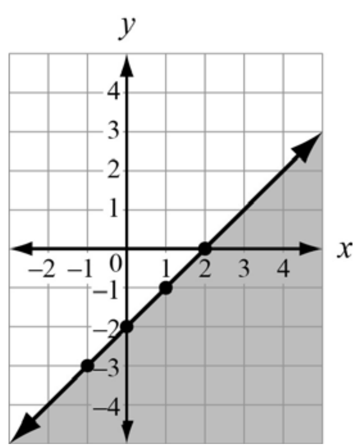
$$\begin{aligned} y + 2 &\leq x \\ y + 2 - 2 &\leq x - 2 \\ y &\leq x - 2 \end{aligned}$$

Create a table of values for the boundary line $y = x - 2$.

x	$x - 2$	y
–1	(–1) – 2	–3
0	(0) – 2	–2
1	(1) – 2	–1
2	(2) – 2	0

Use the points to draw the boundary line. Use a solid line since the inequality uses the ≤ symbol.

Again, use (0, 0) as a test point. Is $0 + 2 \leq 0$? No, so (0, 0) is not a solution of the inequality. Shade the region below the line. The graph for $y + 2 \leq x$ is represented below.



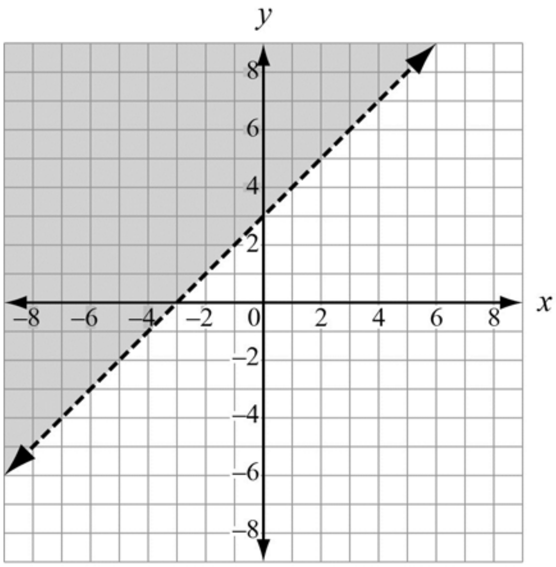
4. A pair of two-variable inequalities can be graphed by finding the region where the shaded half-planes of both inequalities overlap.

Example:

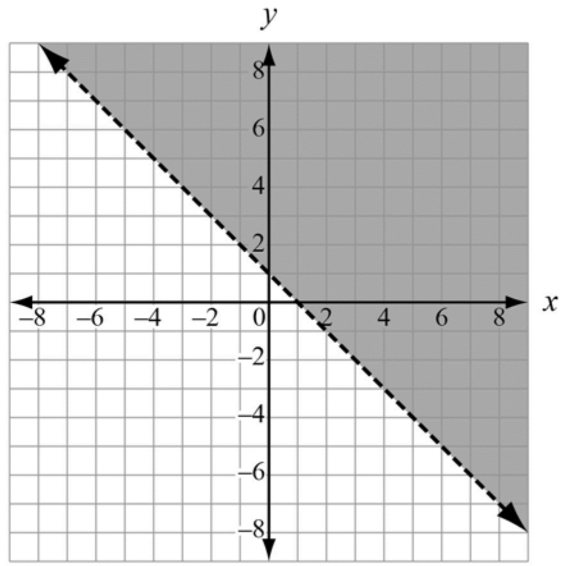
Graph the solution of $y > x + 3$ and $y > -x + 1$.

Solution:

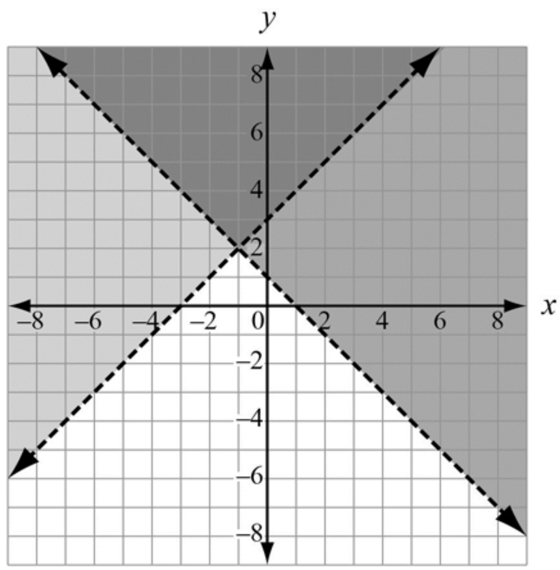
The graph on the left shows $y > x + 3$ and the graph on the right shows $y > -x + 1$. The graph below them shows how the two graphs overlap. The points in the dark, overlapping shading satisfy both inequalities.



$y > x + 3$



$y > -x + 1$



Important Tips

- Know when to use a number line (*one* variable) and when to use a coordinate plane (*two* variables).
- Pay attention to the inequality sign. The ≤ and ≥ symbols mean the boundary is a solution. The < and > symbols mean the boundary is not a solution.

REVIEW EXAMPLE

1) Graph the solution region for $y \leq 2x - 1$.

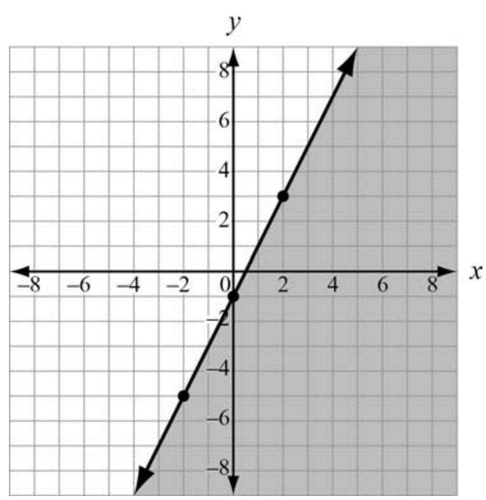
Solution:

First, graph the boundary line, which is solid because the inequality is "less than or equal to." The boundary line would be the equation $y = 2x - 1$. We can make a table of three or more rows and the numbers in the rows can become the coordinates of points on the boundary line.

x	y	(x, y)
–2	2(–2) – 1 = –5	(–2, –5)
0	2(0) – 1 = –1	(0, –1)
2	2(2) – 1 = 3	(2, 3)

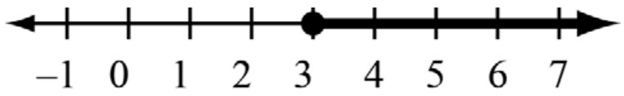
Next, decide which side of the boundary line to shade. Use (0, 0) as a test point.

Is $0 \leq 2(0) - 1$? No, so (0, 0) is not a solution of the inequality. Shade the region below the line. The graph for $y \leq 2x - 1$ is represented below.



EOCT Practice Items

1) Which graph represents $x > 3$?

A.


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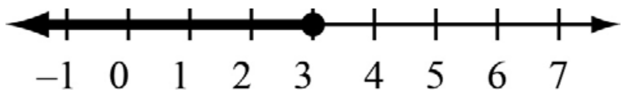

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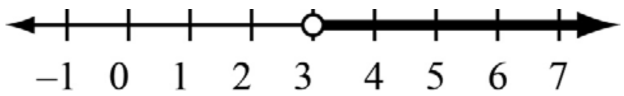
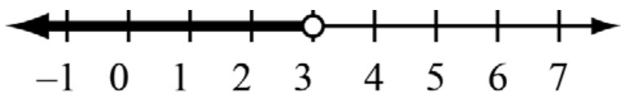
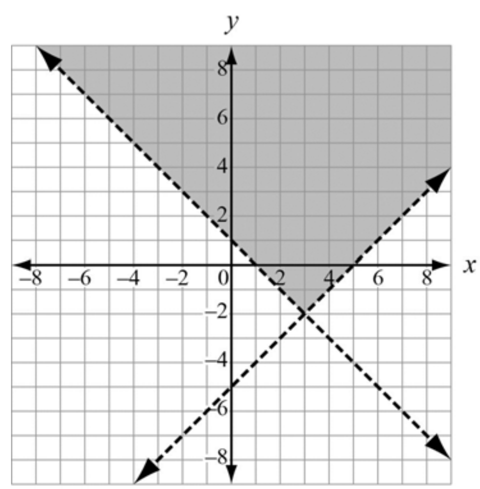


D.


[Key: C]

2) Which pair of inequalities is shown in the graph?



A. $y > -x + 1$ and $y > x - 5$

B. $y > x + 1$ and $y > x - 5$

C. $y > -x + 1$ and $y > -x - 5$

D. $y > x + 1$ and $y > -x - 5$

[Key: A]

Unit 3: Linear and Exponential Functions

In Unit 3, students will learn function notation and develop the concepts of domain and range. They will discover that functions can be combined in ways similar to quantities, such as adding. Students will explore different ways of representing functions (e.g., graphs, rules, tables, and sequences) and interpret functions given graphically, numerically, symbolically, and verbally. Discovering how functions can be transformed, similar to shapes in geometry, and learning about how parameters affect functions are aspects of this unit. Students will also learn how to compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They will interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

KEY STANDARDS

Represent and solve equations and inequalities graphically

MCC9-12.A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). (*Focus on linear and exponential equations and be able to adapt and apply that learning to other types of equations in future courses*.)

MCC9-12.A.REI.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, ~~polynomial, rational, absolute value~~, exponential, ~~and logarithmic~~ functions.★

Understand the concept of a function and use function notation

MCC9-12.F.IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$. (*Draw examples from linear and exponential functions*.)

MCC9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. (*Draw examples from linear and exponential functions*.)

MCC9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$ (*n is greater than or equal to 1*). (*Draw connection to F.BF.2, which requires students to write arithmetic and geometric sequences*.)

Interpret functions that arise in applications in terms of the context

MCC9-12.F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and ~~periodicity~~.★ (*Focus on linear and exponential functions.*)

MCC9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h*(*n*) *gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ (*Focus on linear and exponential functions.*)

MCC9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ (*Focus on linear functions and intervals for exponential functions whose domain is a subset of the integers*.)

Analyze functions using different representations

MCC9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ (*Focus on linear and exponential functions. Include comparisons of two functions presented algebraically*.)

MCC9-12.F.IF.7a Graph linear ~~and quadratic~~ functions and show intercepts, maxima, and minima.★

MCC9-12.F.IF.7e Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, ~~and trigonometric functions, showing period, midline, and amplitude~~.★

MCC9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (*Focus on linear and exponential functions. Include comparisons of two functions presented algebraically*.)

Build a function that models a relationship between two quantities

MCC9-12.F.BF.1 Write a function that describes a relationship between two quantities.★ (*Limit to linear and exponential functions*.)

MCC9-12.F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. (*Limit to linear and exponential functions*.)

MCC9-12.F.BF.1b Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant* *function to a decaying exponential, and relate these functions to the model.* (*Limit to linear and exponential functions*.)

MCC9-12.F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

Build new functions from existing functions

MCC9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (*Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept*.)

Construct and compare linear, ~~quadratic~~, and exponential models and solve problems

MCC9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.★

MCC9-12.F.LE.1a Prove that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals.★

MCC9-12.F.LE.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.★

MCC9-12.F.LE.1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.★

MCC9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

MCC9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, ~~quadratically, or (more generally) as a polynomial function~~.★

Interpret expressions for functions in terms of the situation they model

MCC9-12.F.LE.5 Interpret the parameters in a linear or exponential function in terms of a context. ★ (*Limit exponential functions to those of the form* $f(x) = b^x + k$.)

REPRESENT AND SOLVE EQUATIONS AND INEQUALITIES GRAPHICALLY

KEY IDEAS

1. The graph of a linear equation in two variables is a collection of ordered pair solutions in a coordinate plane. It is a graph of a straight line. Often tables of values are used to organize the ordered pairs.

Example:

Every year Silas buys fudge at the state fair. He buys peanut butter and chocolate. This year he intends to buy \$24 worth of fudge. If chocolate costs \$4 per pound and peanut butter costs \$3 per pound, what are the different combinations of fudge that he can purchase if he only buys whole pounds of fudge?

Solution:

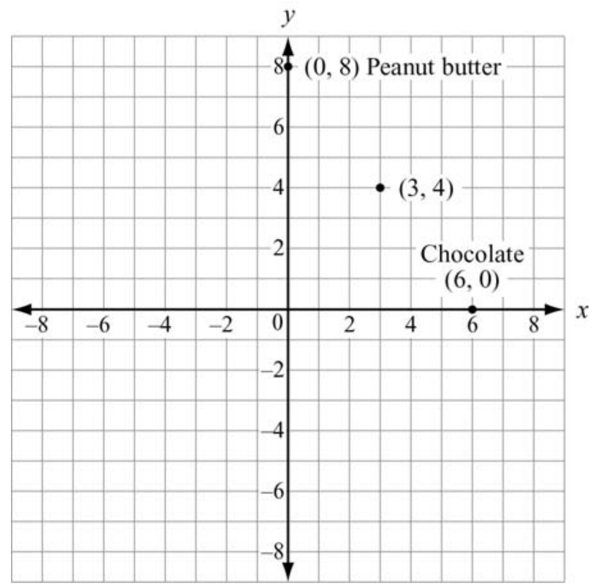
If we let x be the number of pounds of chocolate and y be the number pounds of peanut butter, we can use the equation $4x + 3y = 24$. Now we can solve this equation for y to make it easier to complete our table.

$4x + 3y = 24$	Write the original equation.
$4x - 4x + 3y = 24 - 4x$	Subtract $4x$ from each side.
$3y = 24 - 4x$	Simplify.
$\frac{3y}{3} = \frac{24-4x}{3}$	Divide each side by 3.
$y = \frac{24-4x}{3}$	Simplify.

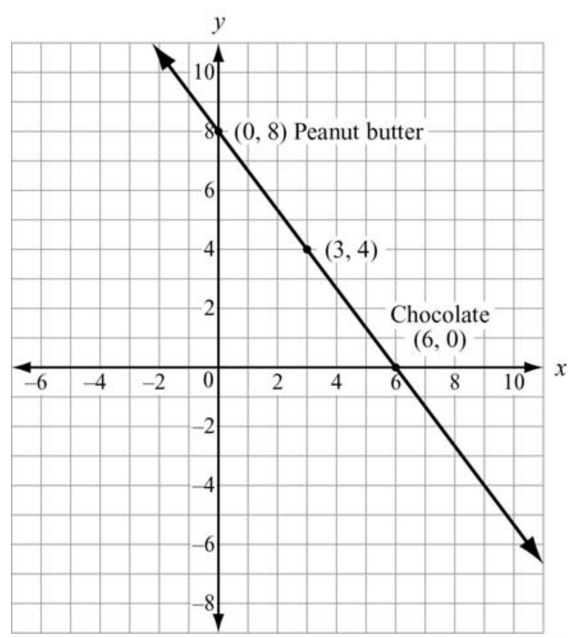
We will only use whole numbers in the table, because Silas will only buy whole pounds of the fudge.

Chocolate, ***x***	**Peanut butter,** ***y***
0	8
1	~~6⅔~~ (not a whole number)
2	~~5⅓~~ (not a whole number)
3	4
4	~~2⅔~~ (not a whole number)
5	~~1⅓~~ (not a whole number)
6	0

The ordered pairs from the table that we want to use are (0, 8), (3, 4), and (6, 0). The graph would look like the one shown below:



Based on the number of points in the graph, there are three possible ways that Silas can buy pounds of the fudge: 8 pounds of peanut butter only, 3 pounds of chocolate and 4 pounds of peanut butter, or 6 pounds of chocolate only. Notice that if the points on the graph were joined, they would form a line. If Silas allowed himself to buy partial pounds of fudge, then there would be many more possible combinations. Each combination would total $24 and be represented by a point on the line that contains (0, 8), (3, 4), and (6, 0).



Example:

Silas decides he does not have to spend exactly \$24 on the fudge, but he will not spend more than \$24. What are the different combinations of fudge purchases he can make?

Solution:

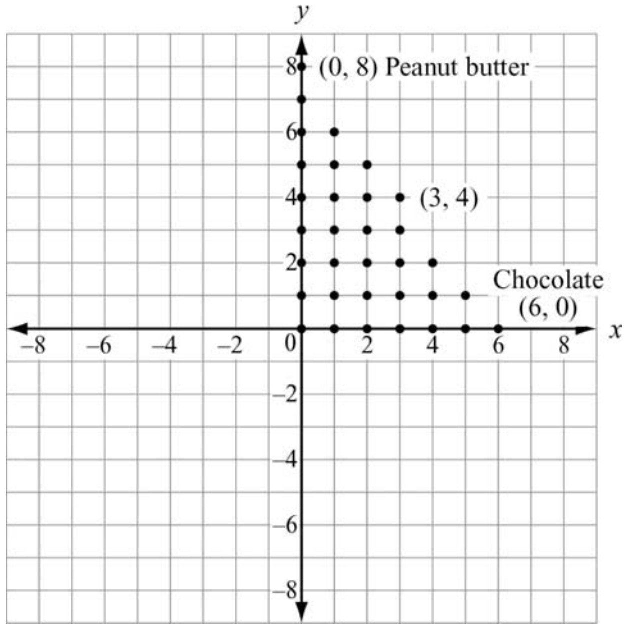
Our new relationship is the inequality $4x + 3y \leq 24$. We'll make a new table of values. We begin by making it easier to find the number of pounds of peanut butter by solving for y.

$4x+3y \leq 24$	Write the original equation.
$4x-4x+3y \leq 24-4x$	Subtract $4x$ from each side.
$3y \leq 24-4x$	Simplify.
$\frac{3y}{3} \leq \frac{24-4x}{3}$	Divide each side by 3.
$y \leq \frac{24-4x}{3}$	Simplify.

Once again, we will only use whole numbers from the table, because Silas will only buy whole pounds of the fudge.

Chocolate, x	**Peanut butter, $y \leq \frac{24-4x}{3}$**
0	0, 1, 2, 3, 4, 5, 6, 7, 8
1	0, 1, 2, 3, 4, 5, 6, $\cancel{6\frac{2}{3}}$
2	0, 1, 2, 3, 4, 5, $\cancel{5\frac{1}{3}}$
3	0, 1, 2, 3, 4
4	0, 1, 2, $\cancel{2\frac{2}{3}}$
5	0, 1, $\cancel{1\frac{1}{3}}$
6	0

Each row in the table has at least one value that is a whole number in both columns. If Silas does not intend to spend all of his \$24, there are many more combinations. Let's look at the graph.



From the points that lie on the graph and the rows in the table, there are 33 combinations of chocolate and peanut butter fudge that Silas can buy. The points on the graph do not look like a line. However, they all appear to fall on or below the line that would join (0, 8), (3, 4), and (6, 0). In fact, the line joining those three points would be the boundary line for the inequality that represents the combinations of pounds of fudge that Silas can purchase. The points do not cover the entire half-plane below that line because it is impossible to purchase negative pounds of fudge. The fact that you cannot purchase negative amounts of fudge puts constraints, or limitations, on the possible x and y values. Both x and y values must be non-negative.

REVIEW EXAMPLES

1) Consider the equations $y = 2x - 3$ and $y = -x + 6$.

a. Complete the tables below, and then graph the equations on the same coordinate axes.

$y = 2x - 3$	
x	y

$y = -x + 6$	
x	y

b. Is there an ordered pair that satisfies both equations? If so, what is it?

c. Graph both equations on the same coordinate plane by plotting the ordered pairs from the tables and connecting the points.

d. Do the lines appear to intersect? If so, where? How can you tell that the point where the lines appear to intersect is a common point for both lines?

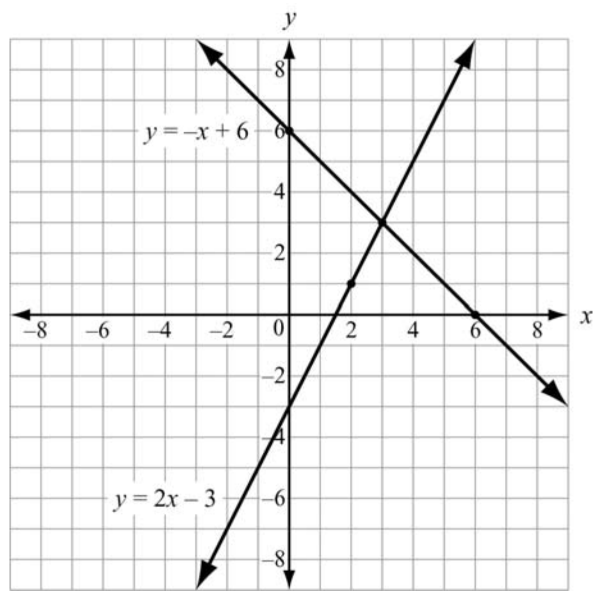
Solution:

a.

$y = 2x - 3$	
x	y
–1	–5
0	–3
1	–1
2	1
3	3

$y = -x + 6$	
x	y
–1	7
0	6
1	5
2	4
3	3

b. Yes, the ordered pair (3, 3) satisfies both equations.

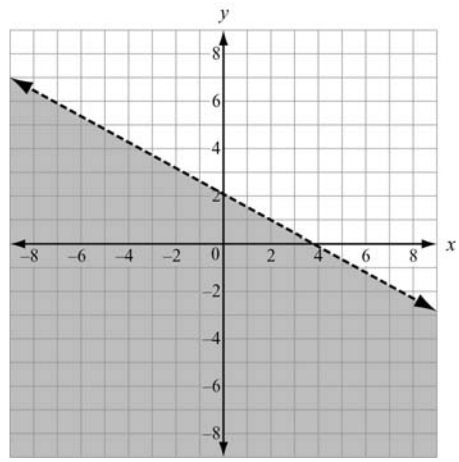
c. 

d. The lines appear to intersect at (3, 3). When $x = 3$ and $y = 3$ are substituted into each equation, the values satisfy both equations. This proves that (3, 3) lies on both lines, which means it is a common solution to both equations.

2) Graph the inequality $x + 2y < 4$.

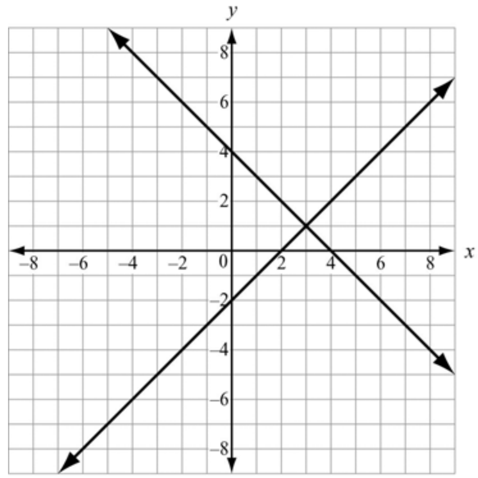
Solution:

The graph looks like a half-plane with a dashed boundary line. The shading is below the line because the points that satisfy the inequality fall below the line.



EOCT Practice Items

1) Two lines are graphed on this coordinate plane.



Which point appears to be a solution of the equations of both lines?

A. (0, –2)

B. (0, 4)

C. (2, 0)

D. (3, 1)

[Key: D]

2) Based on the tables, at what point do the lines $y = -x + 5$ and $y = 2x - 1$ intersect?

$y = -x + 5$	
x	y
–1	6
0	5
1	4
2	3
3	2

$y = 2x - 1$	
x	y
–1	–3
0	–1
1	1
2	3
3	5

A. (1, 1)
B. (3, 5)
C. (2, 3)
D. (3, 2)

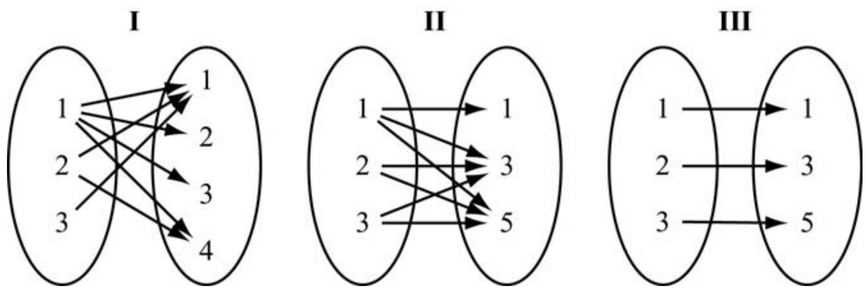
[Key: C]

UNDERSTAND THE CONCEPT OF A FUNCTION AND USE FUNCTION NOTATION

KEY IDEAS

1. There are many ways to show how pairs of quantities are related. Some of them are shown below.

- ***Mapping Diagrams***



- ***Sets of Ordered Pairs***

Set I: {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 4), (3, 1)}

Set II: {(1, 1), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)}

Set III: {(1, 1), (2, 3), (3, 5)}

- ***Tables of Values***

I

x	*y*
1	1
1	2
1	3
1	4
2	1
2	4
3	1

II

x	*y*
1	1
1	3
1	5
2	3
2	5
3	3
3	5

III

x	*y*
1	1
2	3
3	5

The relationship shown in Mapping Diagram I, Set I, and Table I all represent the same paired numbers. Likewise, Mapping II, Set II, and Table II all represent the same quantities. The same goes for the third groups of displays.

Notice the arrows in the mapping diagrams are all arranged from left to right. The numbers on the left side of the mapping diagrams are the same as the x-coordinates in the ordered pairs as well as the values in the first column of the tables. Those numbers are called the input values of a quantitative relationship and are known as the ***domain***. The numbers on the right of the mapping diagrams, the y-coordinates in the ordered pair, and the values in the second column of the table are the output, or ***range***. Every number in the domain is assigned to at least one number of the range.

Mapping diagrams, ordered pairs, and tables of values are good to use when there are a limited number of input and output values. There are some instances when the domain has an infinite number of elements to be assigned. In those cases, it is better to use either an algebraic rule or a graph to show how pairs of values are related. Often we use equations as the algebraic rules for the relationships.

2. A ***function*** is a quantitative relationship where each member of the domain is assigned to exactly one member of the range. Of the relationships on the previous page, only III is a function. In I and II, there were members of the domain that were assigned to two elements of the range. In particular, in I, the value 1 of the domain was paired with 1, 2, 3, and 4 of the range.

Consider the vertical line $x = 2$. Every point on the line has the same x-value and a different y-value. So the value of the domain is paired with infinitely many values of the range. This line is not a function. In fact, all vertical lines are not functions.

3. A function can be described using a ***function rule***, which is an equation that represents an output value, or element of the range, in terms of an input value, or element of the domain.

A function rule can be written in ***function notation***. Here is an example of a function rule and its notation.

$y = 3x + 5$	y is the output and x is the input.
$f(x) = 3x + 5$	Read as "f of x."
$f(2) = 3(2) + 5$	"f of 2" is the output when 2 is the input.

Be careful—do not confuse the parentheses used in notation with multiplication.

Note that all functions have a corresponding graph. The points that lie on the graph of a function are formed using input values, or elements of the domain as the x-coordinates, and output values, or elements of the range as the y-coordinates.

Example:

Given $f(x) = 2x - 1$, find $f(7)$.

Solution:

$f(7) = 2(7) - 1 = 14 - 1 = 13.$

Example:

If $g(6) = 3 - 5(6)$, what is $g(x)$?

Solution:

$g(x) = 3 - 5x$

Example:

If $f(-2) = -4(-2)$, what is $f(b)$?

Solution:

$f(b) = -4b$

Example:

Graph $f(x) = 2x - 1$.

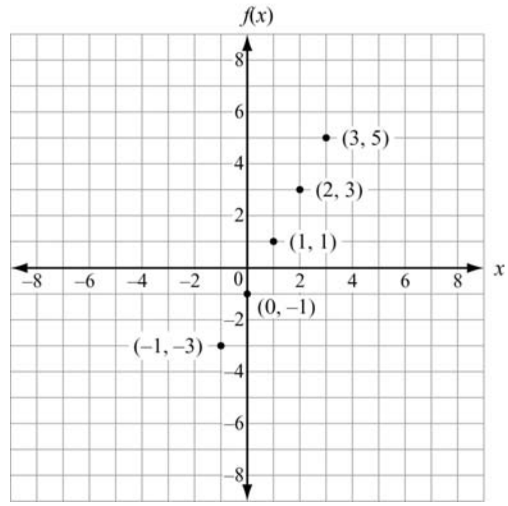
Solution:

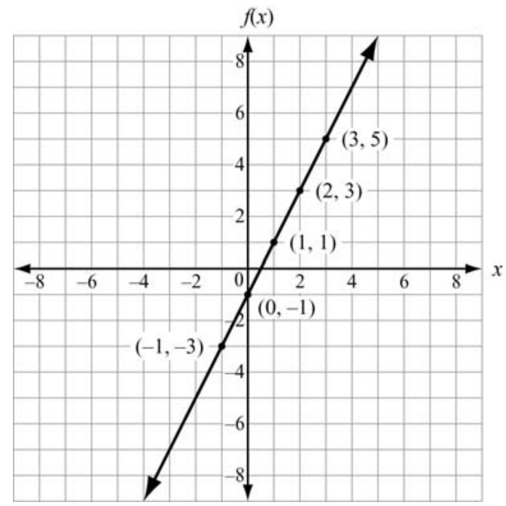
In the function rule $f(x) = 2x - 1$, $f(x)$ is the same as y.

Then we can make a table of x (input) and y (output) values.

x	$f(x)$
–1	–3
0	–1
1	1
2	3
3	5

The values in the rows of the table form ordered pairs. We plot those ordered pairs. If the domain is not specified, we connect the points. If the numbers in the domain are not specified, we assume that it is all real numbers.





4. A ***sequence*** is an ordered list of numbers. Each number in the sequence is called a ***term***. The terms are consecutive or identified as the first term, second term, third term, and so on. The pattern in the sequence is revealed in the relationship between each term and its term number, or in a term's relationship to the previous term in the sequence.

Example:

Consider the sequence: 3, 6, 9, 12, 15, . . . The first term is 3, the second term is 6, the third term is 9, and so on. The ". . ." at the end of the sequence indicates the pattern continues without end. Can this pattern be considered a function?

Solution:

There are different ways of seeing a pattern in the sequence above. One way is to say each number in the sequence is 3 times the number of its term. For example, the fourth term would be 3 times 4, or 12. Looking at the pattern in this way, all you would need to know is the number of the term, and you could predict the value of the term. The value of each term would be a function of its term number. We could use this relationship to write an algebraic rule for the sequence, $y = 3x$, where x is the number of the term and y is the value of the term. This algebraic rule would only assign one number to each input value from the numbers 1, 2, 3, etc. So, we could write a function for the sequence. We can call the function S and write its rule as $S(n) = 3n$, where n is the term number. The domain for the function S would be counting numbers. The range would be the value of the terms in the sequence. When an equation with the term number as a variable is used to describe a sequence, we refer to it as the ***explicit formula*** for the sequence, or the ***closed form***.

Another way to describe the sequence in the example on the previous page is to say each term is three more than the term before it. Instead of using the number of the term, you would need to know a previous term to find a subsequent term's value.

Example:

Consider the sequence: 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. One way to look at this pattern is to say each successive term is half the term before it, and the first term is 16. With this approach you could easily determine the terms for a limited or ***finite*** sequence.

Another way would be to notice that each term is 32 times a power of $\frac{1}{2}$. If n represents the number of the term, each term is 32 times $\frac{1}{2}$ raised to the nth power, or $32 \bullet \left(\frac{1}{2}\right)^n$. This approach lends itself to finding an explicit formula for any missing term if you know its term number. That is, the value of each term would depend on the term number.

Also, the patterns in sequences can be shown by using tables. For example, this table shows the above sequence:

Term Number (*n*)	1	2	3	4	5	6	7	8
Value (a_n)	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Notice the numbers in the top row of the table are consecutive counting numbers starting with one and increasing to the right. The sequence has eight terms, with 16 being the value of the first term and $\frac{1}{8}$ being the value of the eighth term. A sequence with a specific number of terms is finite. If a sequence continues indefinitely, it is called an ***infinite sequence***.

5. If the nth term of a sequence and the common difference between consecutive terms is known, you can find the $(n + 1)$th term using the ***recursive formula*** $a_n = a_{n-1} + d$, where a_n is the *nth* term, n is the number of a term, $n - 1$ is the number of the previous term, and d is the common difference.

Take the sequence 3, 6, 9, 12, 15, . . . as an example. We can find the sixth term of the sequence using the fourth and fifth terms.

The common difference d is $15 - 12 = 3$. So, the sixth term is given by $a_6 = a_5 + 3$.

$a_6 = 15 + 3 = 18$.

Important Tips

- Use language carefully when talking about functions. For example, use f to refer to the function as a whole and use $f(x)$ to refer to the output when the input is x.
- Not all sequences can be represented as functions. Be sure to check all the terms you are provided with before reaching the conclusion that there is a pattern.

REVIEW EXAMPLES

1) A manufacturer keeps track of her monthly costs by using a "cost function" that assigns a total cost for a given number of manufactured items, x. The function is $C(x) = 5{,}000 + 1.3x$.

   a. What is the domain of the function?

   b. What is the cost of 2,000 items?

   c. If costs must be kept below \$10,000 this month, what is the greatest number of items she can manufacture?

Solution:

a. Since x represents a number of manufactured items, it cannot be negative, nor can a fraction of an item be manufactured. Therefore, the domain can only include values that are whole numbers.

b. Substitute 2,000 for x: $C(2{,}000) = 5{,}000 + 1.3(2{,}000) = \$7{,}600$

c. Form an inequality:

$$C(x) < 10{,}000$$
$$5{,}000 + 1.3x < 10{,}000$$
$$1.3x < 5{,}000$$
$$x < 3{,}846.2\text{, or } 3{,}846 \text{ items}$$

2) Consider the first six terms of this sequence: 5, 7, 11, 19, 35, 67, . . .

   a. What is a_1? What is a_3?

   b. If the sequence defines a function, what is the range?

Solution:

a. a_1 is 5 and a_3 is 11.

b. The range is $\{5, 7, 11, 19, 35, 67, \ldots\}$.

3) The function $f(n) = -(1-4n)$ represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

Solution:

n	1	2	3	4	5
$\boldsymbol{f(n)}$	3	7	11	15	19

Since the function is a sequence, the domain would be *n*, the number of each term in the sequence. The set of numbers in the domain can be written as $\{1, 2, 3, 4, 5, \ldots\}$. Notice that the domain is an infinite set of numbers, even though the table only displays the first five elements.

The range is $f(n)$ or (a_n), the output numbers that result from applying the rule $-(1-4n)$. The set of numbers in the range, which is the sequence itself, can be written as {3, 7, 11, 15, 19,…}. This is also an infinite set of numbers, even though the table only displays the first five elements.

EOCT Practice Items

1) The first term in this sequence is –1.

n	1	2	3	4	5	. . .
$\boldsymbol{a_n}$	−1	1	3	5	7	. . .

Which function represents the sequence?

A. $a_n = a_{n-1} + 1$
B. $a_n = a_{n-1} + 2$
C. $a_n = 2a_{n-1} - 1$
D. $a_n = 2a_{n-1} - 3$

[Key: B]

2) Which function is modeled in this table?

x	$f(x)$
1	8
2	11
3	14
4	17

A. $f(x) = x + 7$

B. $f(x) = x + 9$

C. $f(x) = 2x + 5$

D. $f(x) = 3x + 5$

[Key: D]

3) Which explicit formula describes the pattern in this table?

d	C
2	6.28
3	9.42
5	15.70
10	31.40

A. $d = 3.14 \times C$

B. $3.14 \times C = d$

C. $31.4 \times 10 = C$

D. $C = 3.14 \times d$

[Key: D]

4) If $f(12) = 4(12) - 20$, which function gives $f(x)$?

A. $f(x) = 4x$
B. $f(x) = 12x$
C. $f(x) = 4x - 20$
D. $f(x) = 12x - 20$

[Key: C]

INTERPRET FUNCTIONS THAT ARISE IN APPLICATIONS IN TERMS OF THE CONTEXT

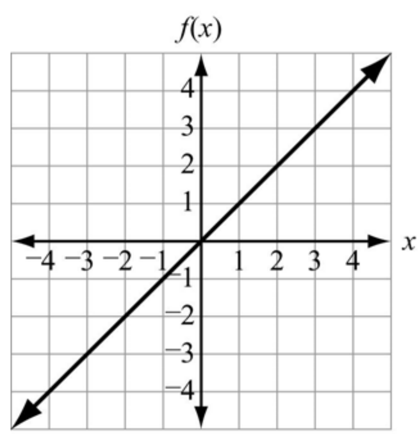
KEY IDEAS

1. Through examining the graph of a function, many of its features are discovered. Features include domain and range, x- and y-intercepts, intervals where the function values are increasing or decreasing, positive or negative, minimums or maximums, and rates of change.

Example:

Consider the graph of $f(x)$. It appears to be an unbroken line and slanted upward.

Linear Function
$f(x) = x$

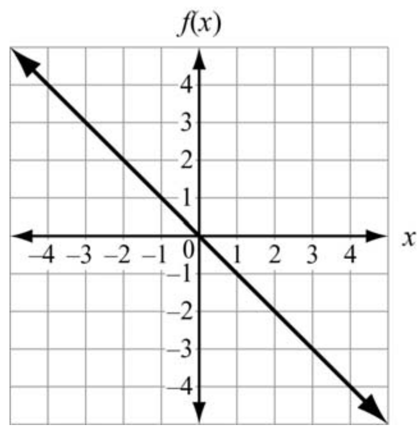


Some of its key features are:

- Domain: All real numbers
- Range: All real numbers
- x-intercept: The line appears to intersect the x-axis at 0
- y-intercept: The line appears to intersect the y-axis at 0
- Increasing: As x increases, $f(x)$ increases
- Decreasing: Never
- Positive: $f(x) > 0$ when $x > 0$
- Negative: $f(x) < 0$ when $x < 0$
- Minimum or Maximum: None
- Rate of change: 1

Example:

Consider the graph of $f(x) = -x$. It appears to be an unbroken line and slanted downward.

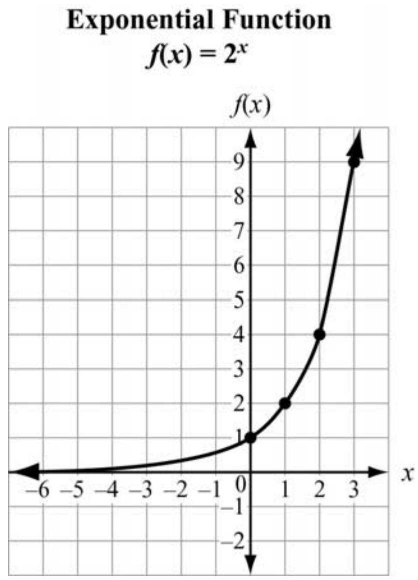


Some of its key features are:

- Domain: All real numbers because there is a point on the graph for every possible x-value
- Range: All real numbers because there is a point on the graph that corresponds to every possible y-value
- x-intercept: It appears to intersect the x-axis at 0
- y-intercept: It appears to intersect the y-axis at 0
- Increasing: The function does not increase
- Decreasing: The graph falls from left to right
- Positive: $f(x)$ is positive for $x < 0$
- Negative: $f(x)$ is negative for $x > 0$
- Minimums or Maximums: None
- Rate of change: –1

Example:

Consider the graph of $f(x) = 2^x$.



Some of its key features are:

- Domain: All real numbers because there is a point on the graph for every possible x-value
- Range: $y > 0$
- x-intercept: None
- y-intercept: It appears to intersect the y-axis at 1
- Increasing: Always
- Decreasing: Never
- Positive: $f(x)$ is positive for all x values
- Negative: $f(x)$ is never negative
- Minimum or Maximum: None
- Rate of change: It appears to vary as the graph has curvature and is not straight

2. Other features of functions can be discovered through examining their tables of values. The intercepts may appear in a table of values. From the differences of $f(x)$ values over various intervals we can tell if a function grows at a constant rate of change.

Example:

Let $h(x)$ be the number of person-hours it takes to assemble x engines in a factory. The company's accountant determines that the time it takes depends on start-up time and the number of engines to be completed. It takes 6.5 hours to set up the machinery to make the engines and about 5.25 hours to completely assemble one. The relationship is modeled with the function $h(x) = 6.5 + 5.25x$. Next, he makes a table of values to check his function against his production records. He starts with 0 engines because of the startup time.

The realistic domain for the accountant's function would be whole numbers, because you cannot manufacture a negative number of engines.

***x*, engines**	***h*(*x*), hours**
0	6.5
1	11.75
2	17
3	22.25
4	27.5
5	32.75
10	59
100	531.5

From the table we can see the y-intercept. The y-intercept is the y-value when $x = 0$. The very first row of the table indicates the y-intercept is 6.5. Since we do not see a number 0 in the $h(x)$ column, we cannot tell from the table if there is an x-intercept. The x-intercept is the value when $h(x) = 0$.

$$\begin{aligned} h(x) &= 6.5 + 5.25x \\ 0 &= 6.5 + 5.25x \\ -6.5 &= 5.25x \\ -1.24 &= x \end{aligned}$$

The x-value when $y = 0$ is negative, which is not possible in the context of this example.

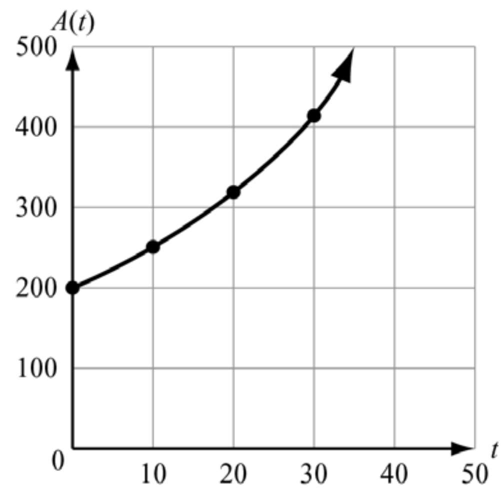
The accountant's table also gives us an idea of the rate of change of the function. We should notice that when x values are increasing by 1, the $h(x)$ values are growing by increments of 5.25. There appears to be a constant rate of change when the input values increase by the same amount. The increase from both 3 engines to 4 engines and 4 engines to 5 engines is 5.25 hours. The average rate of change can be calculated by comparing the values in the first and last rows of the table. The increase in number of engines made is 100 – 0, or 100. The increase in hours is 531.5 – 6.5, or 525. The average rate of change is $\frac{525}{100} = 5.25.$ The units for this average rate of change would be hours/engine, which happens to be the exact amount of time it takes to make an engine.

Important Tips

- Begin exploration of a new function by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.
- You cannot always find exact values from a graph. Always check your answers using the equation.

REVIEW EXAMPLES

1) The amount accumulated in a bank account over a time period t and based on an initial deposit of \$200 is found using the formula $A(t) = 200(1.025)^t$, $t \geq 0$. Time, t, is represented on the horizontal axis. The accumulated amount, $A(t)$, is represented on the vertical axis.



a. What are the intercepts of the function?
b. What is the domain of the function?
c. Why are all the t values non-negative?
d. What is the range of the function?
e. Does the function have a maximum or minimum value?

Solution:

a. There is no t-intercept. The function crosses the vertical axis at 200.
b. The domain is $t \geq 0$.
c. The t values are all non-negative because they represent time, and time cannot be negative.
d. $A(t) \geq 200$.
e. The function has no maximum value. Its minimum value is 200.

2) A company uses the function $V(x) = 28{,}000 - 1{,}750x$ to represent the depreciation of a truck, where $V(x)$ is the value of the truck and x is the number of years after its purchase. Use the table of values shown below.

x, years	**$V(x)$, value in $**
0	28,000
1	26,250
2	24,500
3	22,750
4	21,000
5	19,250

a. What is the y-intercept of the graph of the function?

b. Does the graph of the function have an x-intercept?

c. Does the function increase or decrease?

Solution:

a. From the table, when $x = 0$, $V(x) = 28{,}000$. So, the y-intercept is 28,000.

b. Yes, it does have an x-intercept, although it is not shown in the table. The x-intercept is the value of x when $V(x) = 0$.

$$\begin{aligned} 0 &= 28{,}000 - 1{,}750x \\ -28{,}000 &= -1{,}750x \\ 16 &= x \end{aligned}$$

The x-intercept is 16.

c. As $x > 0$, $V(x)$ decreases. Therefore, the function decreases.

EOCT Practice Items

1) A farmer owns a horse that can continuously run an average of 8 miles an hour for up to 6 hours. Let *y* be the distance the horse can travel for a given *x* amount of time in hours. The horse's progress can be modeled by a function.

Which of the following describes the domain of the function?

A. $0 \le x \le 6$

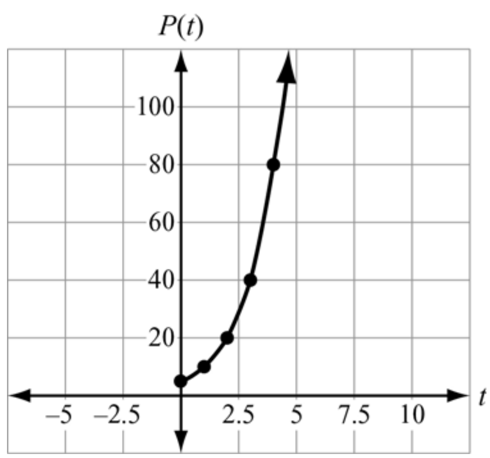
B. $0 \le y \le 6$

C. $0 \le x \le 48$

D. $0 \le y \le 48$

[Key: A]

2) A population of squirrels doubles every year. Initially there were 5 squirrels. A biologist studying the squirrels created a function to model their population growth, $P(t) = 5(2^t)$ where *t* is time. The graph of the function is shown. What is the range of the function?



A. any real number

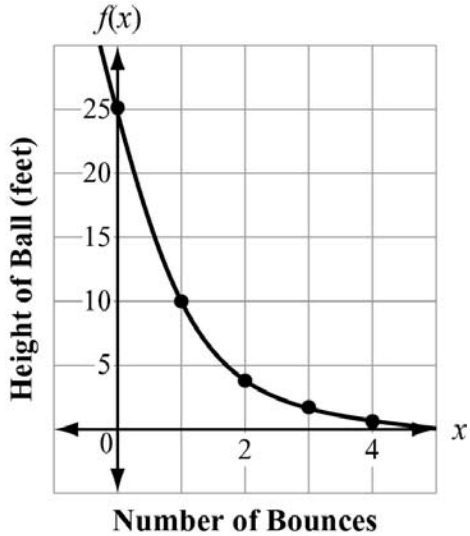
B. any whole number greater than 0

C. any whole number greater than 5

D. any whole number greater than or equal to 5

[Key: D]

3) The function graphed on this coordinate grid shows *f*(*x*), the height of a dropped ball in feet after its *x*th bounce.



On which bounce was the height of the ball 10 feet?

A. bounce 1

B. bounce 2

C. bounce 3

D. bounce 4

[Key: A]

ANALYZE FUNCTIONS USING DIFFERENT REPRESENTATIONS

KEY IDEAS

1. When working with functions, it is essential to be able to interpret the specific quantitative relationship regardless of the manner of its presentation. Understanding different representations of functions such as tables, graphs, and equations makes interpreting relationships between quantities easier. Beginning with lines, we will learn how each representation aids our understanding of a function. Almost all lines are functions, except vertical lines, because they assign multiple elements of their range to just one element in their domain. All linear functions can be written in the form $f(x) = ax + b$, where a and b are real numbers and x is a variable to which the function f assigns a corresponding value, $f(x)$.

Example:

Consider the linear functions $f(x) = x + 5$, $g(x) = 2x – 5$, and $h(x) = –2x$.

First we will make a table of values for each equation. To begin, we need to decide on the domains. In theory, $f(x)$, $g(x)$, and $h(x)$ can accept any number as input. So, the three of them have all real numbers as their domains. But, for a table we can only include a few elements of their domains. We should choose a sample that includes negative numbers, 0, and positive numbers. Place the elements of the domain in the left column, usually in ascending order. Then apply the function's assignment rule to determine the corresponding element in the range. Place it in the right column.

x	$f(x) = x + 5$
–3	2
–2	3
–1	4
0	5
1	6
2	7
3	8
4	9

x	$g(x) = 2x – 5$
–3	–11
–2	–9
–1	–7
0	–5
1	–3
2	–1
3	1
4	3

x	$h(x) = –2x$
–3	6
–2	4
–1	2
0	0
1	–2
2	–4
3	–6
4	–8

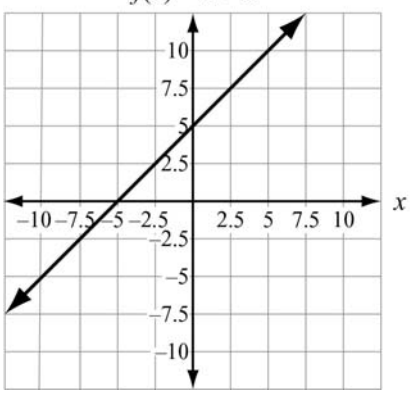
We can note several features about the functions just from their table of values.

- $f(x)$ has a y-intercept of 5. When x is 0, $f(x) = 5$. It is represented by (0, 5) on its graph.
- $g(x)$ has a y-intercept of –5. When x is 0, $g(x) = –5$. It is represented by (0, –5) on its graph.

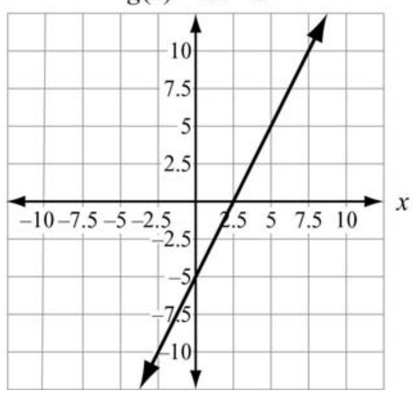
- $h(x)$ has a y-intercept of 0. When x is 0, $h(x) = 0$. It is represented by (0, 0) on its graph.
- $h(x)$ has a x-intercept of 0. When $h(x) = 0$, $x = 0$. It is represented by (0, 0) on its graph.
- $f(x)$ has an average rate of change of 1. $\dfrac{9-2}{4-(-3)} = 1$
- $g(x)$ has an average rate of change of 2. $\dfrac{3-(-11)}{4-(-3)} = 2$
- $h(x)$ has an average rate of change of –2. $\dfrac{(-8)-6}{4-(-3)} = -2$

Now we will take a look at the graphs of $f(x)$, $g(x)$, and $h(x)$.

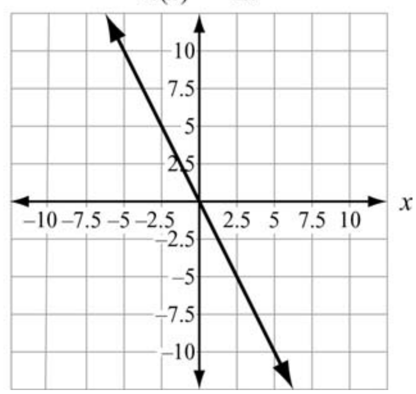
$f(x) = x + 5$



$g(x) = 2x - 5$



$h(x) = -2x$



Their graphs confirm what we already learned about their intercepts and their constant rates of change. The graphs suggest other information:

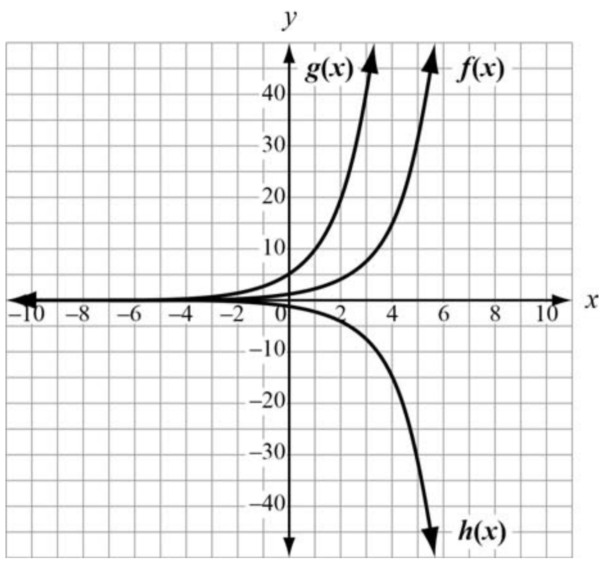
- $f(x)$ appears to have positive values for $x > -5$ and negative values for $x < -5$
- $f(x)$ appears to be always increasing with no maximum or minimum values
- $g(x)$ appears to have positive values for $x > 2.5$ and negative values for $x < 2.5$
- $g(x)$ appears to be always increasing with no maximum or minimum values
- $h(x)$ appears to have positive values for $x < 0$ and negative values for $x > 0$
- $h(x)$ appears to be always decreasing with no maximum or minimum values

To confirm these observations we can work with the equations for the functions. We suspect $f(x)$ is positive for $x > -5$. Since $f(x)$ is positive whenever $f(x) > 0$, write and solve the inequality $x + 5 > 0$ and solve for x. We get $f(x) > 0$ when $x > -5$. We can confirm all our observations about $f(x)$ from working with the equation. Likewise, the observations about $g(x)$ and $h(x)$ can be confirmed using their equations.

2. The three ways of representing a function also apply to exponential functions. Exponential functions are built using powers. A power is the combination of a base with an exponent. For example, in the power 5^3, the base is 5 and the exponent is 3. A function with a power where the exponent is a variable is an exponential function. Exponential functions are of the form $f(x) = ab^x$ where $a \neq 0$, with $b > 0$, and $b \neq 1$. In an exponential function, the base b is a constant.

Example:

Consider $f(x) = 2^x$, $g(x) = 5 \cdot 2^x$, and $h(x) = -2^x$. For all three functions, $f(x)$, $g(x)$, and $h(x)$, the base is 2. The value of the coefficient causes the graphs to look different.



From the graphs, you can make the following observations:

- $f(x)$ appears to have a y-intercept at 1.
- $g(x)$ appears to have a y-intercept at 5.
- $h(x)$ appears to have a y-intercept at –1.
- For $f(x)$, as x increases, $f(x)$ increases and as x decreases, $f(x)$ approaches 0.
- For $g(x)$, as x increases, $g(x)$ increases and as x decreases, $g(x)$ approaches 0.
- For $h(x)$, as x increases, $h(x)$ decreases and as x decreases, $h(x)$ approaches 0.
- None of the functions appear to have a constant rate of change.

Now look at tables of values for these functions.

x	$f(x) = 2^x$
–3	$\frac{1}{8}$
–2	$\frac{1}{4}$
–1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16

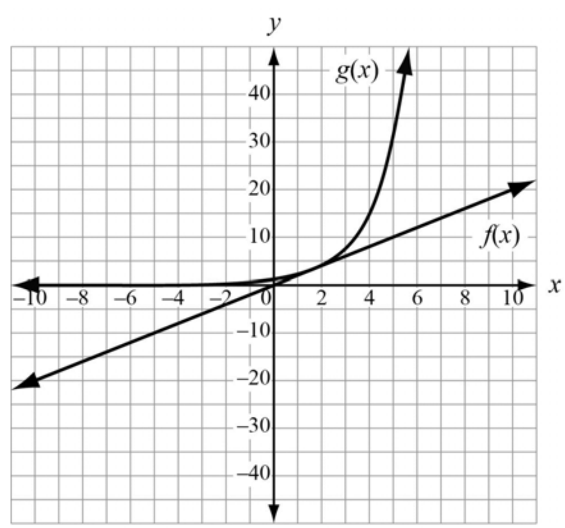
x	$g(x) = 5 \cdot 2^x$
–3	$\frac{5}{8}$
–2	$\frac{5}{4}$
–1	$\frac{5}{2}$
0	5
1	10
2	20
3	40
4	80

x	$h(x) = -2^x$
–3	$-\frac{1}{8}$
–2	$-\frac{1}{4}$
–1	$-\frac{1}{2}$
0	–1
1	–2
2	–4
3	–8
4	–16

The tables confirm all three functions have y-intercepts: $f(0) = 1$, $g(0) = 5$, and $h(0) = -1$. Although the tables do not show a constant rate of change for any of the functions, a rate of change can be determined on a specific interval.

3. Comparing functions helps us gain a better understanding of them. Let's take a look at a linear function and the graph of an exponential function.

Consider $f(x) = 2x$ and the graph of $g(x)$ below. The function $f(x) = 2x$ represents a linear relationship. This graph shows an exponential relationship. We know the linear function has a graph that is a straight line and has a constant rate of change. The graph of the exponential function is curved and has a varying rate of change. Both curves will have y-intercepts. The exponential curve appears to have a y-intercept below 5. The y-intercept of the line is $f(0) = 2 \cdot 0 = 0$. Since $f(0) = 0$, the line must pass through the point (0, 0), so 0 must also be the x-intercept of the line. The graph of the exponential function does not appear to have an x-intercept, though the curve appears to come very close to the x-axis.



For $f(x) = 2x$, the domain and range are all real numbers. For the exponential function, the domain is defined for all real numbers and the range is defined for positive values.

So, while the two graphs share some features, they also have significant differences; the most important of which is that the linear function is a straight line and has a constant rate of change, while the exponential function is curved with a rate of change that is increasing.

Important Tips

- Remember the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function.
- Be familiar with important features of a function such as intercepts, domain, range, minimum and maximums, and periods of increasing and decreasing values.

REVIEW EXAMPLES

1) What are the key features of the function $p(x) = \frac{1}{2}x - 3$?

Solution:

First, notice that the function is linear. The domain for the function is the possible numbers we can substitute for x. Since the function is linear, the domain is all real numbers. The graphic representation will give us a better idea of its range.

We can determine the y-intercept by finding $p(0)$:

$$p(0) = \frac{1}{2}(0) - 3 = -3$$

So, the graph of $p(x)$ will intersect the y-axis at (0, –3). To find the x-intercept, we have to solve the equation $p(x) = 0$.

$$\frac{1}{2}x - 3 = 0$$
$$\frac{1}{2}x = 3$$
$$x = 6$$

So, the x-intercept is 6. The line intersects the x-axis at (6, 0).

Now we will make a table of values to investigate the rate of change of $f(x)$.

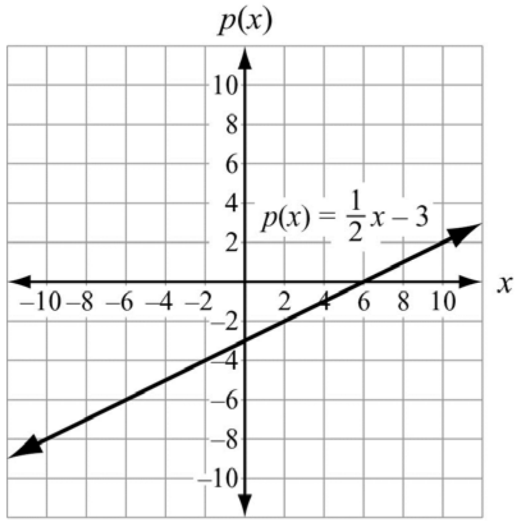
x	$p(x)=\frac{1}{2}x-3$
–3	$-\frac{9}{2}$
–2	– 4
–1	$-\frac{7}{2}$
0	–3
1	$-\frac{5}{2}$
2	–2
3	$-\frac{3}{2}$
4	–1

Notice the row that contains the values 0 and –3. These numbers correspond to the point where the line intersects the y-axis, confirming that the y-intercept is –3. Since 0 does not appear in the right column, the coordinates of the x-intercept point are not in the table of values. We notice that the values in the right column keep increasing by $\frac{1}{2}$. We can calculate the average rate of change.

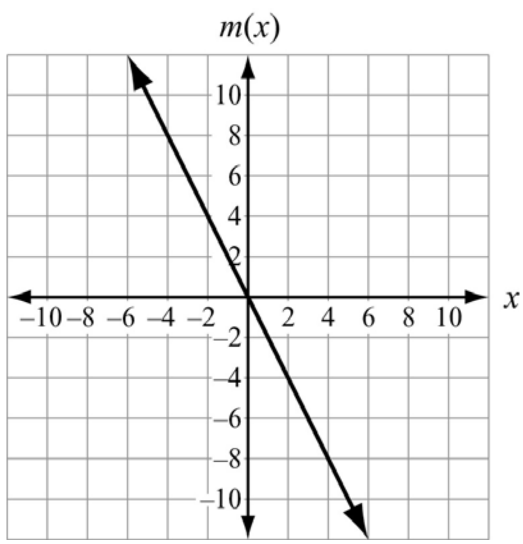
Average rate of change: $\dfrac{-1-\frac{-9}{2}}{4-(-3)}=\dfrac{1}{2}$

It turns out the average rate of change is the same as the incremental differences in the outputs. This confirms the function $p(x)$ has a constant rate of change. Notice that $\frac{1}{2}$ is the coefficient of x in the function rule.

Now we will examine the graph. The graph shows a line that appears to be always increasing. Since the line has no minimum or maximum value, its range would be all real numbers. The function appears to have positive values for $x > 6$ and negative values for $x < 6$.



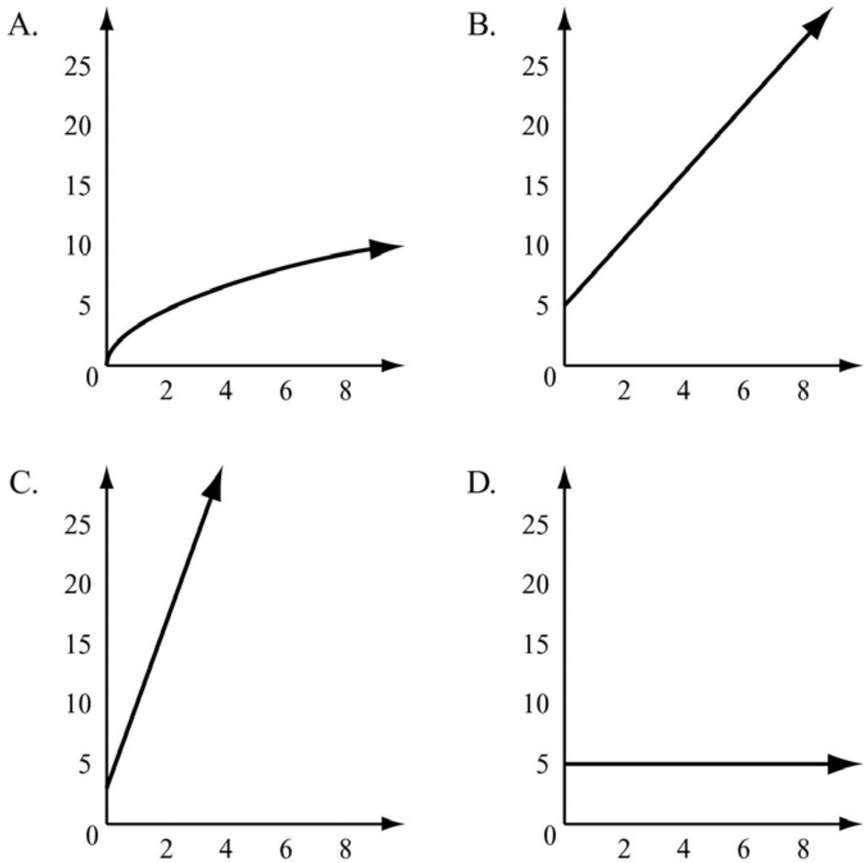
2) Compare $p(x) = \frac{1}{2}x - 3$ from the previous example with the function $m(x)$ in the graph below.



The graph of $m(x)$ intersects both the x- and y-axes at 0. It appears to have a domain of all real numbers and a range of all real numbers. So, $m(x)$ and $p(x)$ have the same domain and range. The graph appears to have a constant rate of change and is decreasing. It has positive values when $x < 0$ and negative values when $x > 0$.

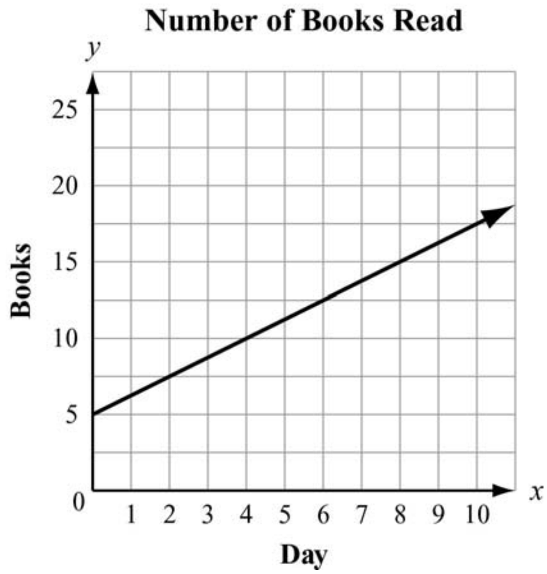
EOCT Practice Items

1) To rent a canoe, the cost is $3 for the oars and life preserver, plus $5 an hour for the canoe. Which graph models the cost of renting a canoe?



[Key: C]

2) Juan and Patti decided to see who could read the most books in a month. They began to keep track after Patti had already read 5 books that month. This graph shows the number of books Patti read for the next 10 days.



If Juan has read no books before the fourth day of the month and he reads at the same rate as Patti, how many books will he have read by day 12?

A. 5

B. 10

C. 15

D. 20

[Key: B]

BUILD A FUNCTION THAT MODELS A RELATIONSHIP BETWEEN TWO QUANTITIES

KEY IDEAS

1. Modeling a quantitative relationship can be a challenge. But there are some techniques we can use to make modeling easier. Mostly, we decide which kind of model to use based on the rate of change of the function's values.

Example:

Joe started with \$13. He has been saving \$2 each week to purchase a baseball glove. The amount of money Joe has depends on how many weeks he has been saving. So, the number of weeks and the amount Joe has saved are related. We can begin with the function $S(x)$, where S is the amount he has saved and x is the number of weeks. Since we know that he started with \$13 and that he saves \$2 each week we can use a linear model, one where the change is constant.

A linear model for a function is $f(x) = ax + b$, where a and b are any real numbers and x is the independent variable.

So, the model is $S(x) = 2x + 13$ which will generate the amount Joe has saved after x weeks.

Example:

Pete withdraws half his savings every week. If he started with \$400, can a rule be written for how much Pete has left each week? We know the amount Pete has left depends on the week. Once again we can start with the amount Pete has, $A(x)$. The amount depends on the week number, x. However, this time the rate of change is not constant. Therefore, the previous method for finding a function will not work. We could set up the model as

$A(x) = 400 \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2}$ and use $\frac{1}{2}$ as the number of weeks, x.

Or, we can use a power of $\frac{1}{2}$:

$$A(x) = 400 \cdot \left(\frac{1}{2}\right)^x$$

Note that the function assumes Pete had \$400 at week 0 and withdraws half during week 1. The exponential function will generate the amount Pete has after x weeks.

2. Sometimes the data for a function is presented as a sequence.

Example:

Suppose we know the total number of cookies eaten by Rachel on a day-to-day basis over the course of a week. We might get a sequence like this: 3, 5, 7, 9, 11, 13, 15. There are two ways we could model this sequence. The first would be the explicit way. We would arrange the sequence in a table. Note that the symbol Δ in the third row means change or difference.

n	1	2	3	4	5	6	7
a_n	3	5	7	9	11	13	15
Δa_n	---------	5 – 3 = 2	7 – 5 = 2	9 – 7 = 2	11 – 9 = 2	13 – 11 = 2	15 – 13 = 2

Since the difference between successive terms of the sequence is constant, namely 2, we can again use a linear model. But this time we do not know the y-intercept because there is no zero term ($n = 0$). However, if we work backward, a_0—the term before the first—would be 1, so the starting number would be 1. That leaves us with an explicit formula: $f(n) = 2n + 1$, for $n > 0$ (n is an integer). A sequence that can be modeled with a linear function is called an ***arithmetic sequence***.

Another way to look at the sequence is recursively. We need to express term n (a_n) in terms of a previous term. Since the constant difference is 2, we know $a_n = a_{n-1} + 2$ for $n > 1$, with $a_1 = 3$.

3. Some sequences can be modeled exponentially. For a sequence to fit an exponential model, the ratio of successive terms is constant.

Example:

Consider the number of sit-ups Clara does each week as listed in the sequence 3, 6, 12, 24, 48, 96, 192. Clara is doing twice as many sit-ups each successive week. It might be easier to put the sequence in a table to analyze it.

n	1	2	3	4	5	6	7
a_n	3	6	12	24	48	96	192
$a_n \div a_{n-1}$	-----	6 ÷ 3 = 2	12 ÷ 6 = 2	24 ÷ 12 = 2	48 ÷ 24 = 2	96 ÷ 48 = 2	192 ÷ 96 = 2

It appears as if each term is twice the term before it. But the difference between the terms is not constant. This type of sequence shows exponential growth. The function type is $f(x) = a(b^x)$. In this type of function, b is the coefficient and b^x is the growth power. For the sequence above, the growth power is 2^x because the terms keep doubling. To find b you need to know the first term. The first term is 3. The second term is the first term of the sequence multiplied by the common ratio once. The third term is the first term multiplied by the common ratio twice. Since that pattern continues, our exponential function is $f(x) = 3(2^{x-1})$. The function $f(x) = 3(2^{x-1})$ would be the explicit or ***closed form*** for the sequence. A sequence that can be modeled by an exponential function is a ***geometric sequence***.

The sequence could also have a recursive rule. Since the next term is twice the previous term, the recursive rule would be $a_n = 2 \cdot a_{n-1}$, with a first term, a_1, of 3.

4. Exponential functions have lots of practical uses. They are used in many real-life situations.

Example:

A scientist collects data on a colony of microbes. She notes these numbers:

Day	1	2	3	4	5	6
a_n	800	400	200	100	50	25
$a_n \div a_{n-1}$	----	$400 \div 800 = 0.5$	$200 \div 400 = 0.5$	$100 \div 200 = 0.5$	$50 \div 100 = 0.5$	$25 \div 50 = 0.5$

Since the ratio between successive terms is a constant 0.5, she believes the growth power is (0.5^{x-1}), where x is the number of the day. She uses 800 for the coefficient, since that was the population on the first day. The function she used to model the population size was $f(x) = 800(0.5^{x-1})$.

Important Tips

- Examine function values to draw conclusions about the rate of change.
- Keep in mind the general forms of a linear function and exponential function.

REVIEW EXAMPLES

1) The terms of a sequence increase by a constant amount. If the first term is 7 and the fourth term is 16:

   a. List the first six terms of the sequence.

   b. What is the explicit formula for the sequence?

   c. What is the recursive rule for the sequence?

Solution:

a. The sequence would be: 7, 10, 13, 16, 19, 22, . . . $\frac{16-7}{4-1} = \frac{9}{3} = 3.$ If the difference between the first and fourth terms is 9, the constant difference is 3. So, the sequence is arithmetic.

b. Since the constant difference is 3, $a = 3$. Because the first term is 7, $b = 7 - 3 = 4$. So, the explicit formula is: $f(n) = 3(n) + 4$, for $n > 0$.

c. Since the difference between successive terms is 3, $a_n = a_{n-1} + 3$ with $a_1 = 7$.

EOCT Practice Items

1) Which function represents this sequence?

n	1	2	3	4	5	…
a_n	6	18	54	162	486	…

A. $f(n) = 3^{n-1}$
B. $f(n) = 6^{n-1}$
C. $f(n) = 3(6^{n-1})$
D. $f(n) = 6(3^{n-1})$

[Key: D]

2) The first term in this sequence is 3.

n	1	2	3	4	5	. . .
a_n	3	10	17	24	31	. . .

Which function represents the sequence?

A. $f(n) = n + 3$
B. $f(n) = 7n - 4$
C. $f(n) = 3n + 7$
D. $f(n) = n + 7$

[Key: B]

3) The points (0, 1), (1, 5), (2, 25), (3, 125) are on the graph of a function. Which equation represents that function?

A. $f(x) = 2^x$
B. $f(x) = 3^x$
C. $f(x) = 4^x$
D. $f(x) = 5^x$

[Key: D]

BUILD NEW FUNCTIONS FROM EXISTING FUNCTIONS

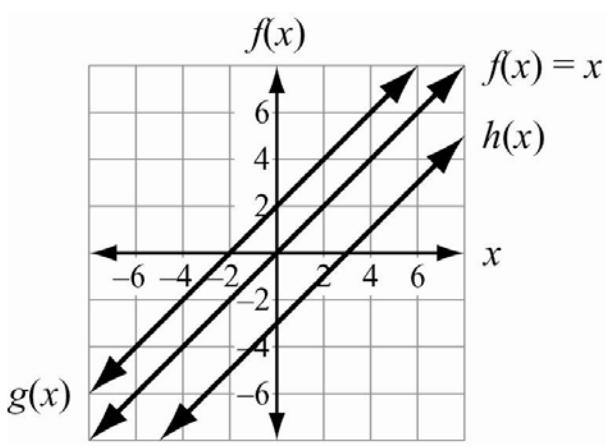
KEY IDEAS

1. Functions can be transformed in many ways. Whenever a function rule is transformed, the transformation affects the graph. One way to transform a function is to increase or decrease the assigned values by a specified amount. Such adjustments have the effect of shifting the function's graph up or down. These shifts are called ***translations***. The original curve is moved from one place to another on the coordinate plane the same as translations in geometry.

Example:

If $f(x) = x$, how will $g(x) = f(x) + 2$ and $h(x) = f(x) - 3$ compare?

Solution:



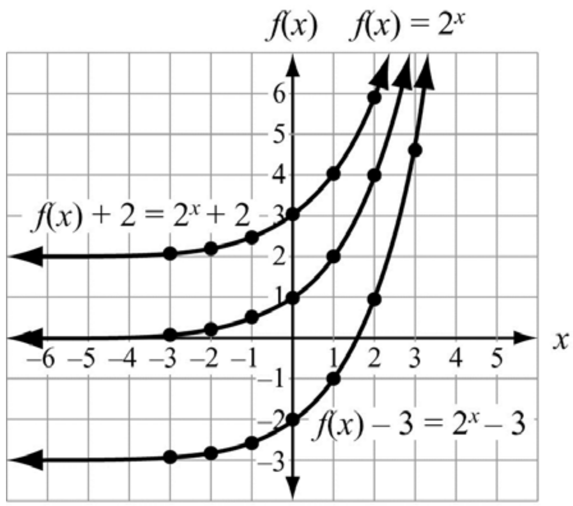
As expected, the lines all have the same shape, rate of change, domain, and range. The adjustments affected the intercepts. When 2 is added to the function's values, the y-intercept is 2 units higher. When 3 is subtracted, the y-intercept drops 3 units. When quantities are added or subtracted from a function's values, it causes a vertical translation of the graph. This effect would be true of all types of functions.

Example:

If $f(x) = 2^x$, how will $f(x) + 2$ and $f(x) - 3$ compare?

Solution:

By using substitution, it makes sense that if $f(x) = 2^x$, then $f(x) + b = 2^x + b$. We will compare graphs of $f(x) = 2^x$, $f(x) + 2 = 2^x + 2$, and $f(x) - 3 = 2^x - 3$.



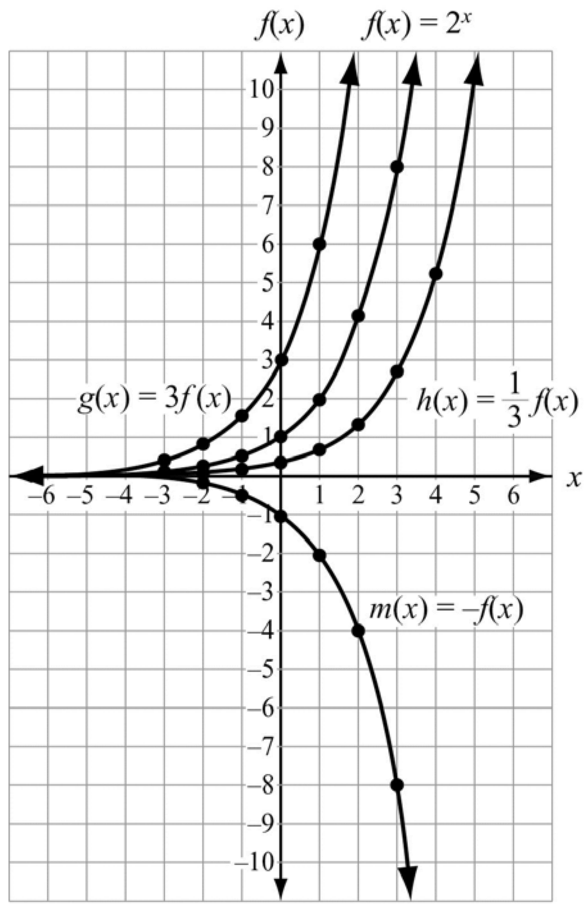
The curves have not changed shape. Their domains and ranges are unchanged. However, the curves are shifted vertically. The function $f(x) + 2 = 2^x + 2$ is a translation of $f(x) = 2^x$ upward by 2 units. The function $f(x) - 3 = 2^x - 3$ is a translation downward by 3 units.

2. Functions can be adjusted by factors as well as sums or subtractions. The factors can affect the functions either before or after they make their assignments. When a function is multiplied by a factor after the value is assigned, it stretches or shrinks the graph of the function. If the factor is greater than 1, it stretches the graph of the function. If the factor is between 0 and 1, it shrinks the graph of the function. If the factor is –1, it reflects the function over the x-axis.

Example:

If $f(x) = 2^x$, how will $g(x) = 3f(x)$, $h(x) = \frac{1}{3}f(x)$ and $m(x) = -f(x)$ compare?

Solution:



The graphs all have y-intercepts, but the rates of change are affected.

In summary:

- ✓ Adjustments made by adding or subtracting values, either before or after the function, assign values to inputs and cause translations of the graphs.
- ✓ Adjustments made by using factors, either before or after the function, assign values to inputs and affect the rates of change of the functions and their graphs.
- ✓ Multiplying by a factor of –1 reflects a linear function over the x-axis.

4. We call f an even function if $f(x) = f(-x)$ for all values in its domain.

Example:

Suppose f is an even function and the point (2, 7) is on the graph of f. Name one other point that must be on the graph of f.

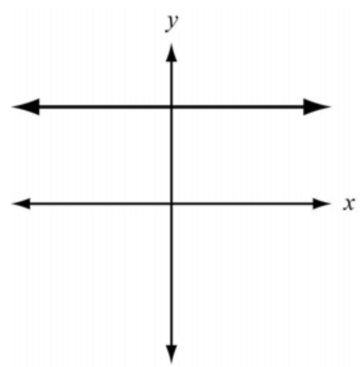
Solution:

Since (2, 7) is on the graph, 2 is in the domain and $f(2) = 7$. By definition of an even function, $f(-2) = f(2) = 7$.

Therefore, (−2, 7) is also on the graph of f.

5. The graph of an ***even function*** has line symmetry with respect to the y-axis.

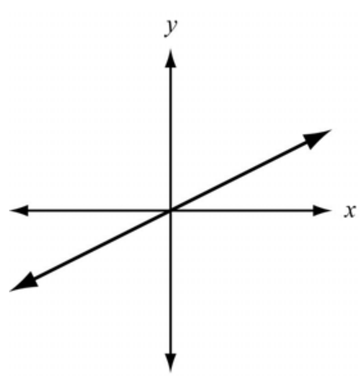
This is a graph of an even function.



Notice that, for any number b, the points (x, b) and $(-x, b)$ are at the same height on the grid and are equidistant from the y-axis. That means they represent line symmetry with respect to the y-axis.

6. We call f an ***odd function*** if $f(-x) = -f(x)$ for all values in its domain.

This is a graph of an odd function.



The graph of an odd function has rotational symmetry of 180° about the origin. This is also called symmetry with respect to the origin. Whenever the graph of an odd function contains the point (a, b) it also contains the point $(-a, -b)$.

Example:

Suppose f is an odd function and the point $(-2, 8)$ is on the graph of f. Name one other point that must be on the graph of f.

Solution:

Since $(-2, 8)$ is on the graph, -2 is in the domain and $f(-2) = 8$. By definition of an odd function, $-(-2)$, or 2, is also in the domain and $f(2) = -f(-2) = -8$.

Therefore, $(2, -8)$ is also on the graph of f.

Important Tip

- Graph transformed functions in the same coordinate plane to see how their graphs compare.

REVIEW EXAMPLES

1) For the function $f(x) = 3^x$:

   a. Find the function that represents a 5 unit translation upward of the function.

   b. Is the function even, odd, or neither even nor odd?

Solution:

a. $f(x) = 3^x + 5$
b. $f(x)$ is neither even nor odd. It has no symmetry.

2) Given the function $f(x) = 3x + 4$:

a. Compare $f(x)$ to $3f(x)$.

b. Compare $f(x)$ to $f(3x)$.

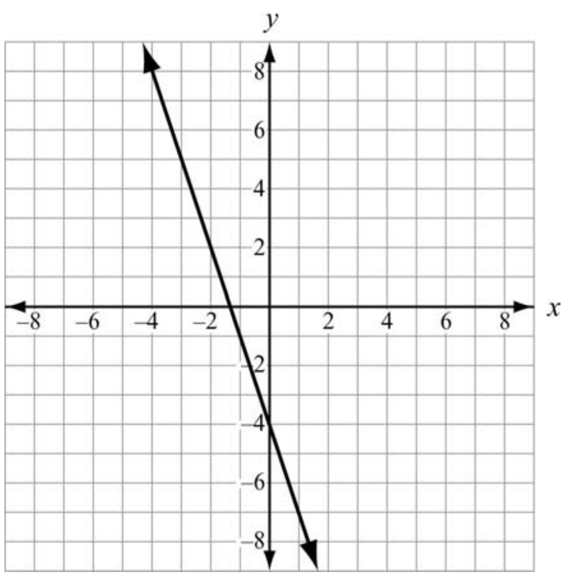
c. Draw the graph of $-f(x)$.

d. Which has the fastest growth rate: $f(x)$, $3f(x)$, or $-f(x)$?

Solution:

a. $3f(x) = 3(3x + 4) = 9x + 12$. So, it crosses the y-axis 8 units higher and has 3 times the growth rate of $f(x)$.

b. $f(3x) = 3(3x) + 4 = 9x + 4$. So, it has the same intercept as $f(x)$ and 3 times the growth rate.

c. 

d. $3f(x)$

EOCT Practice Items

1) A function g is an odd function. If $g(-3) = 4$, which other point lies on the graph of g?

A. (3, –4)
B. (–3, –4)
C. (4, –3)
D. (–4, 3)

[Key: A]

2) Which statement is true about the function $f(x) = 7$?

A. The function is odd because $-f(x) = f(-x)$.
B. The function is even because $-f(x) = f(-x)$.
C. The function is odd because $f(x) = f(-x)$.
D. The function is even because $f(x) = f(-x)$.

[Key: D]

CONSTRUCT AND COMPARE LINEAR, ~~QUADRATIC~~, AND EXPONENTIAL MODELS AND SOLVE PROBLEMS

KEY IDEAS

1. Recognizing linear and exponential growth rates is key to modeling a quantitative relationship. The most common growth rates in nature are either linear or exponential. Linear growth happens when the dependent variable changes are the same for equal intervals of the independent variable. Exponential growth happens when the dependent variable changes at the same percent rate for equal intervals of the independent variable.

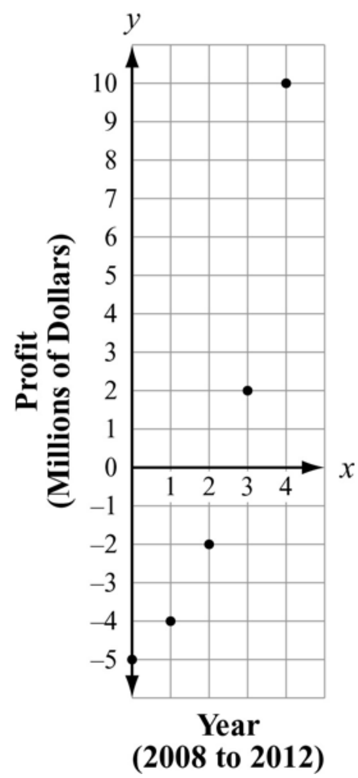
   Example:

   Given a table of values, look for a constant rate of change in the y, or $f(x)$, column. The table below shows a constant rate of change, namely –2, in the $f(x)$ column for each unit change in the independent variable x. The table also shows the y-intercept of the relation. The function has a y-intercept of +1, the $f(0)$ value. These two pieces of information allow us to find a model for the relationship. When the change in $f(x)$ is constant, we use a linear model, $f(x) = ax + b$, where a represents the constant rate of change and b the y-intercept. For the given table, the a value is –2, the constant change in the $f(x)$ values, and b is the $f(x)$ value of 1. The function is $f(x) = -2x + 1$. Using the linear model, we are looking for an explicit formula for the function.

x	$f(x)$	Change in $f(x)$
–2	5	------------
–1	3	3 – 5 = –2
0	1	1 – 3 = –2
1	–1	–1 – 1 = –2
2	–3	–3 – (–1) = –2

Example:

Given the graph below, compare the coordinates of points to determine if there is either linear or exponential growth.



The points represent the profit/loss of a new company over its first five years, from 2008 to 2012. The company started out $5,000,000 in debt. After five years it had a profit of $10,000,000. From the arrangement of the points, the pattern does not look linear. We can check by considering the coordinates of the points and using a table of values.

x	***y***	**Change in *y***
0	–5,000,000	------------
1	–4,000,000	–4,000,000 – (–5,000,000) = 1,000,000
2	–2,000,000	–2,000,000 – (–4,000,000) = 2,000,000
3	2,000,000	2,000,000 – (–2,000,000) = 4,000,000
4	10,000,000	10,000,000 – (2,000,000) = 8,000,000

The *y* changes are not constant for equal *x* intervals. However, the ratios of successive differences are equal.

$$\frac{2,000,000}{1,000,000} = \frac{4,000,000}{2,000,000} = \frac{2}{1} = 2$$

Having a constant percent for the growth rate for equal intervals indicates exponential growth. The relationship can be modeled using an exponential function. However, our example does not cross the y-axis at 1, or 1,000,000. Since the initial profit value was not \$1,000,000, the exponential function has been translated downward. The amount of the translation is \$6,000,000. We model the company's growth as:

$$P(x) = 1{,}000{,}000(2^x) - 6{,}000{,}000$$

2. We can use our analysis tools to compare ***growth rates***. For example, it might be interesting to consider whether you would like your pay raises to be linear or exponential. Linear growth is characterized by a constant number. With a linear growth, a value grows by the same amount each time. Exponential growth is characterized by a percent which is called the growth rate.

Example:

Suppose you start work and earn \$600 per week. After one year, you are given two choices for getting a raise: a) 2% per year, or b) a flat \$15 per week raise for each successive year. Which option is better? We can make a table with both options and see what happens.

Year	Weekly Pay	
	2% per year raise	**\$15 per week raise**
0	\$600	\$600
1	\$612	\$615
2	\$624.24	\$630
3	\$636.72	\$645

Looking at years 1 through 3, the \$15 per week option seems better. However, look closely at the 2% column. Though the pay increases start out smaller each year, they are growing exponentially. For some year in the future, the 2% increase in salary will be more than the \$15 per week increase in salary. If you know the number of years you expect to work at the company, it will help determine which option is best.

Important Tips

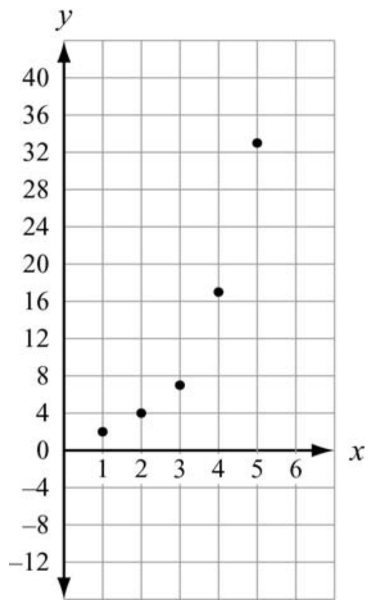
- Examine function values carefully.
- Remember that a linear function has a constant rate of change.
- Keep in mind that growth rates are modeled with exponential functions.

REVIEW EXAMPLES

1) The swans on Elsworth Pond have been increasing in number each year. Felix has been keeping track and so far he has counted 2, 4, 7, 17, and 33 swans each year for the past five years.

   a. Make a scatter plot of the swan population.
   b. What type of model would be a better fit, linear or exponential? Explain your answer.
   c. How many swans should Felix expect next year if the trend continues? Explain your answer.

Solution:

a. 

b. Exponential; the growth rate is not constant. The swan population appears to be nearly doubling every year.
c. There could be about 64 swans next year. A function modeling the swan growth would be $P(x) = 2^x$, which would predict $P(6) = 2^6 = 64$.

2) Given the sequence 7, 10, 13, 16, . . .

   a. Does it appear to be linear or exponential?
   b. Determine a function to describe the sequence.
   c. What would the 20th term of the sequence be?

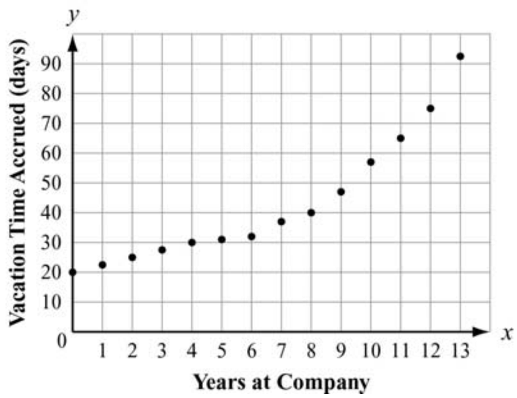
Solution:

a. Linear; the terms increase by a constant amount, 3.
b. $f(x) = 3x + 4$. The growth rate is 3 and the first term is 4 more than 3 times 1.
c. 64; $f(20) = 3(20) + 4 = 64$

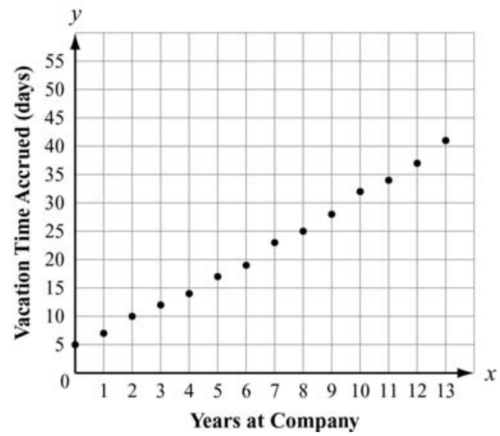
EOCT Practice Items

1) Which scatter plot BEST represents a model of linear growth?

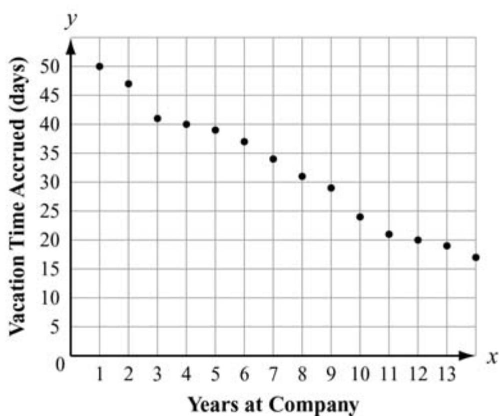
A.



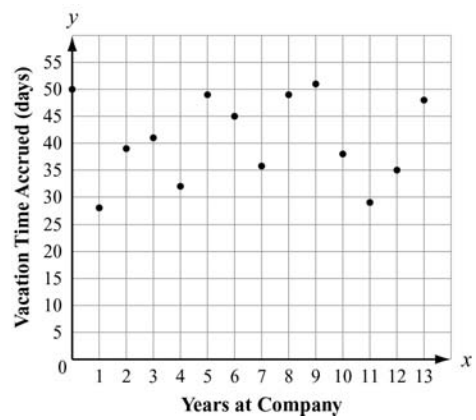
B.



C.

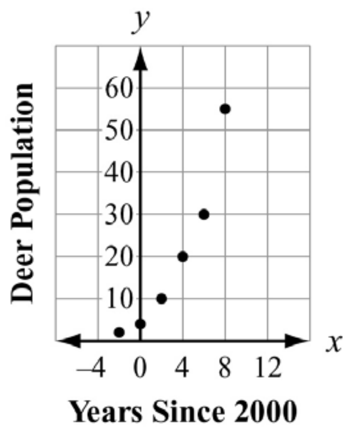


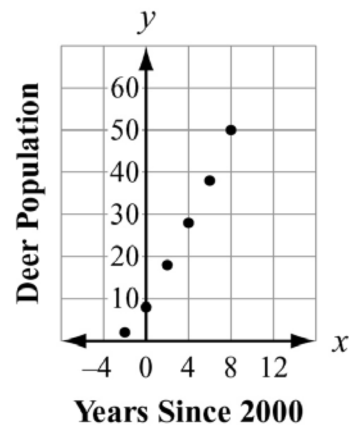
D.

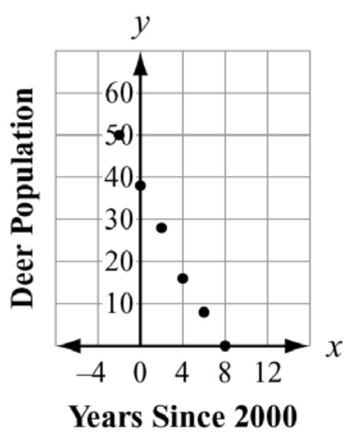


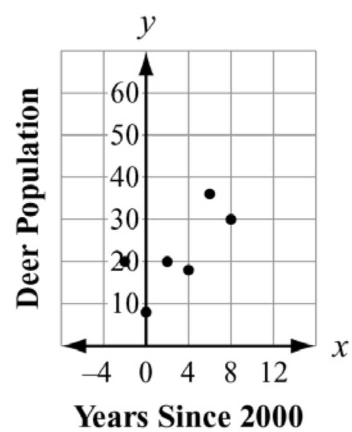
[Key: B]

2) Which scatter plot BEST represents a model of exponential growth?

A. 

B. 

C. 

D. 

[Key: A]

3) Which table represents an exponential function?

A.

x	0	1	2	3	4
y	5	6	7	8	9

B.

x	0	1	2	3	4
y	0	22	44	66	88

C.

x	0	1	2	3	4
y	5	13	21	29	37

D.

x	0	1	2	3	4
y	0	3	9	27	81

[Key: D]

INTERPRET EXPRESSIONS FOR FUNCTIONS IN TERMS OF THE SITUATION THEY MODEL

KEY IDEAS

1. A ***parameter*** is a coefficient or a constant term in the equation that affects the behavior of the function. Though parameters may be expressed as letters when a relationship is generalized, they are not variables. A parameter as a constant term generally affects the intercepts of a function. If the parameter is a coefficient, in general it will affect the rate of change. Below are several examples of specific parameters.

Equation	**Parameter(s)**
$y = 3x + 5$	coefficient 3, constant 5
$f(x) = \frac{9}{5}x + 32$	coefficient $\frac{9}{5}$, constant 32
$v(t) = v_0 + at$	coefficient a, constant v_0
$y = mx + b$	coefficient m, constant b

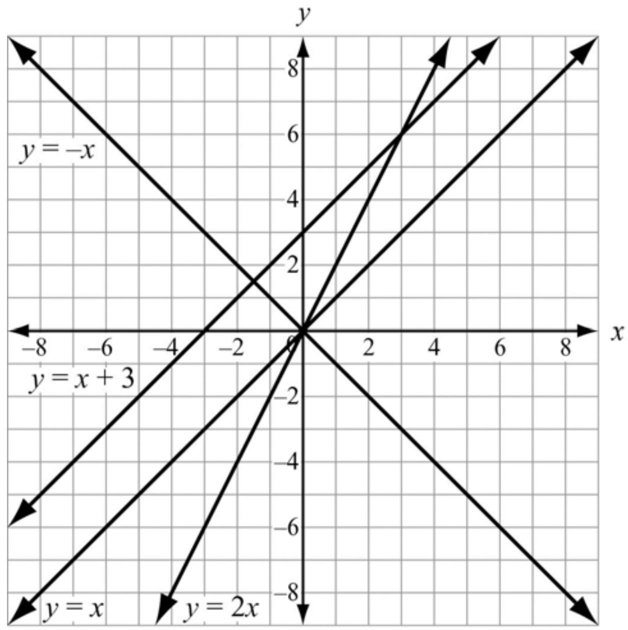
We can look at the effect of parameters on a linear function.

Example:

Consider the lines $y = x$, $y = 2x$, $y = -x$, and $y = x + 3$. The coefficients of x are parameters. The +3 in the last equation is a parameter. We can make one table for all four lines and then compare their graphs.

x	***y***			
	$y = x$	$y = 2x$	$y = -x$	$y = x + 3$
–3	–3	–6	3	0
–2	–2	–4	2	1
–1	–1	–2	1	2
0	0	0	0	3
1	1	2	–1	4
2	2	4	–2	5
3	3	6	–3	6
4	4	8	–4	7

The four linear graphs show the effects of the parameters.



- Only $y = x + 3$ has a different y- intercept. The $+3$ translated the $y = x$ graph up 3 units.
- Both $y = x$ and $y = x + 3$ have the same slope (rate of change). The coefficients of the x terms are both 1.
- The lines $y = -x$ and $y = 2x$ have different slopes than $y = x$. The coefficients of the x terms, –1 and 2, affect the slopes of the lines.
- The line $y = -x$ is the reflection of $y = x$ over the x-axis. It is the only line with a negative slope.
- The rate of change of $y = 2x$ is twice that of $y = x$.

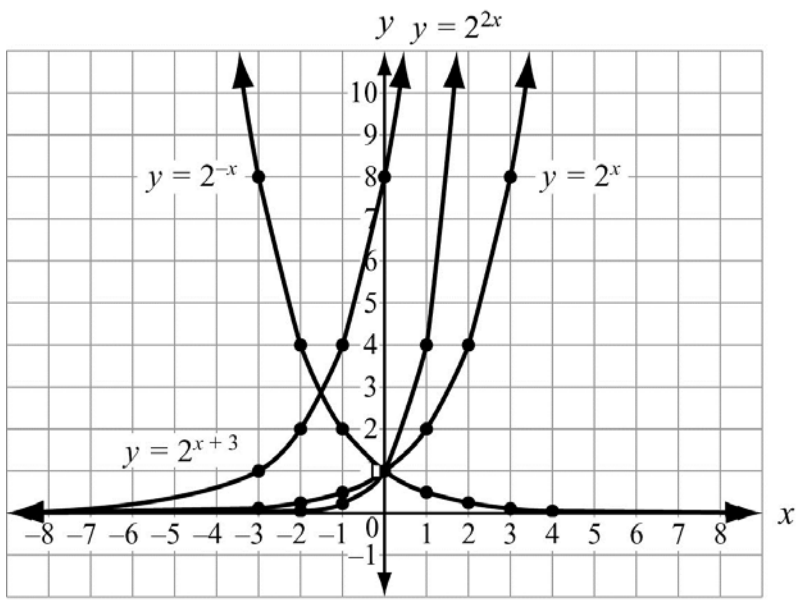
2. We can look at the effect of parameters on an exponential function, in particular when applied to the independent variable, not the base.

Example:

Consider the exponential curves $y = 2^x$, $y = 2^{-x}$, $y = 2^{3x}$, and $y = 2^{x+3}$. The coefficients of the exponent x are parameters. The $+3$ applied to the exponent x in the last equation is a parameter. We can make one table for all four exponentials and then compare the effects.

- $y = 2^{-x}$ is a mirror image of $y = 2^{x}$ with the y-axis as mirror. It has the same y-intercept.
- $y = 2^{2x}$ has the same y-intercept as $y = 2^{x}$, but rises much more steeply.
- $y = 2^{x+3}$ is the $y = 2^{x}$ curve translated 3 units to the left.

x	**y**			
	$y = 2^{x}$	$y = 2^{-x}$	$y = 2^{2x}$	$y = 2^{x+3}$
–3	$\frac{1}{8}$	8	$\frac{1}{64}$	1
–2	$\frac{1}{4}$	4	$\frac{1}{16}$	2
–1	$\frac{1}{2}$	2	$\frac{1}{4}$	4
0	1	1	1	8
1	2	$\frac{1}{2}$	4	16
2	4	$\frac{1}{4}$	16	32
3	8	$\frac{1}{8}$	64	64
4	16	$\frac{1}{16}$	256	128



3. Parameters show up in equations when there is a parent function. Parameters affect the shape and position of the parent function. When we determine a function that models a specific set of data, we are often called upon to find the parent function's parameters.

Example:

Katherine has heard that you can estimate the outside temperature from the number of times a cricket chirps. It turns out that the warmer it is outside the more a cricket will chirp. She has these three pieces of information:

- ✓ a cricket chirps 76 times a minute at 56° (76, 56)
- ✓ a cricket chirps 212 times per minute at 90° (212, 90)
- ✓ the relationship is linear

Estimate the function.

Solution:

The basic linear model or parent function is $f(x) = mx + b$, where m is the slope of the line and b is the y-intercept.

So, the slope, or rate of change, is one of our parameters. First we will determine the constant rate of change, called the slope, m.

$$m = \frac{90-56}{212-76} = \frac{34}{136} = \frac{1}{4}$$

Since we now know that $f(x) = \frac{1}{4}x + b,$ we can substitute in one of our ordered pairs to determine b.

$$T(76) = 56\text{, so } \frac{1}{4}(76) + b = 56$$

$$19 + b = 56$$

$$19 + b - 19 = 56 - 19$$

$$b = 37$$

Our parameters are $m = \frac{1}{4}$ and $b = 37$.

Our function for the temperature is $T(c) = \frac{1}{4}c + 37.$

REVIEW EXAMPLES

1) Alice finds her flower bulbs multiply each year. She started with just 24 tulip plants. After one year she had 72 plants. Two years later she had 120. Find a linear function to model the growth of Alice's bulbs.

Solution:

The data points are (0, 24), (1, 72), (2, 120). The linear model is $B(p) = m(p) + b$.

We know $b = 24$ because $B(0) = 24$ and $B(0)$ gives the vertical intercept.

Find m: $m = \frac{120-72}{2-1} = \frac{48}{1} = 48.$

The parameters are $m = 48$ and $b = 24$.

The function modeling the growth of the bulbs is $B(p) = 48p + 24$.

2) Suppose Alice discovers she counted wrong the second year and she actually had 216 tulip plants. She realizes the growth is not linear because the rate of change was not the same. She must use an exponential model for the growth of her tulip bulbs. Find the exponential function to model the growth.

Solution:

We now have the points (0, 24), (1, 72), (2, 216). We use a parent exponential model:

$$B(p) = a(b^x)$$

In the exponential model the parameter a would be the initial number. So, $a = 24$. To find the base b, we substitute a coordinate pair into the parent function.

$$B(1) = 72\text{, so } 24(b^1) = 72,\ b^1 = \frac{72}{24} = 3\text{, so } b = 3.$$

Now we have the parameter and the base. The exponential model for Alice's bulbs would be:

$$B(p) = 24(3^p).$$

EOCT Practice Item

1) If the parent function is $f(x) = mx + b$, what is the value of the parameter m for the line passing through the points (–2, 7) and (4, 3)?

A. -9

B. $-\frac{3}{2}$

C. -2

D. $-\frac{2}{3}$

[Key: D]

Unit 4: Describing Data

In this unit, students will learn informative ways to display both categorical and quantitative data. They will learn ways of interpreting those displays and pitfalls to avoid when presented with data. Among the methods they will study are two-way frequency charts for categorical data and lines-of-best-fit for quantitative data. Measures of central tendency will be revisited along with measures of spread.

KEY STANDARDS

Summarize, represent, and interpret data on a single count or measurable variable

MCC9-12.S.ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).★

MCC9-12.S.ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, ~~standard deviation~~) of two or more different data sets.★ (*Standard deviation is left for Advanced Algebra, use MAD as a measure of spread*.)

MCC9-12.S.ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).★

Summarize, represent, and interpret data on two categorical and quantitative variables

MCC9-12.S.ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.★

MCC9-12.S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★

MCC9-12.S.ID.6a Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, ~~quadratic~~, and exponential models.★

MCC9-12.S.ID.6b Informally assess the fit of a function by plotting and analyzing residuals.★

MCC9-12.S.ID.6c Fit a linear function for a scatter plot that suggests a linear association.★

Interpret linear models

MCC9-12.S.ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.★

MCC9-12.S.ID.8 Compute (using technology) and interpret the correlation coefficient of a linear fit.★

MCC9-12.S.ID.9 Distinguish between correlation and causation.★

SUMMARIZE, REPRESENT, AND INTERPRET DATA ON A SINGLE COUNT OR MEASURABLE VARIABLE

KEY IDEAS

1. Two ***measures of central tendency*** that help describe a data set are mean and median.
   - The ***mean*** is the sum of the data values divided by the total number of data values.
   - The ***median*** is the middle value when the data values are written in order from least to greatest. If a data set has an even number of data values, the median is the mean of the two middle values.

2. The ***first quartile*** or the ***lower quartile***, Q_1, is the median of the lower half of a data set.

   Example:

   Ray's scores on his mathematics tests were 70, 85, 78, 90, 84, 82, and 83. To find the first quartile of his scores, write them in order from least to greatest.

   70, 78, 82, 83, 84, 85, 90

   The scores in the lower half of the data set are 70, 78, and 82. The median of the lower half of the scores is 78.

   So, the first quartile is 78.

3. The ***third quartile*** or the ***upper quartile***, Q_3, is the median of the upper half of a data set.

   Example:

   Referring to the previous example, the upper half of Ray's scores is 84, 85, and 90. The median of the upper half of the scores is 85.

   So, the third quartile is 85.

4. The ***interquartile range*** **(IQR)** of a data set is the difference between the third and first quartiles, or $Q_3 - Q_1$.

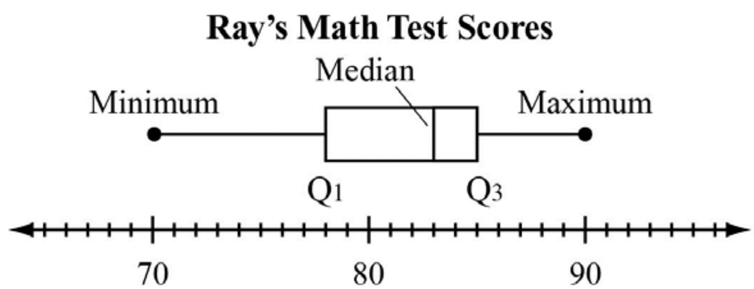
Example:

Referring again to the example of Ray's scores, to find the interquartile range subtract the first quartile from the third quartile. The interquartile range of Ray's scores is 85 − 78 = 7.

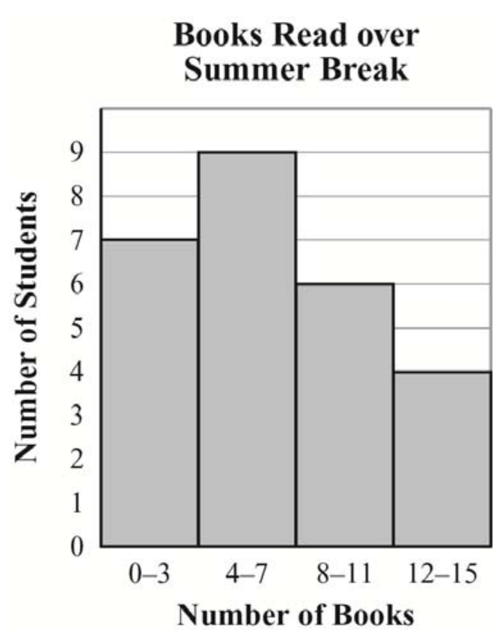
5. The most common displays for quantitative data are dot plots, histograms, box plots, and frequency distributions. A ***box plot*** is a diagram used to display a data set that uses quartiles to form the center box and the minimum and maximum to form the whiskers.

Example:

For the data in Key Idea 2, the box plot would look like the one shown below:



A ***histogram*** is a graphical display that subdivides the data into class intervals, called ***bins***, and uses a rectangle to show the frequency of observations in those intervals—for example, you might use intervals of 0–3, 4–7, 8–11, and 12–15 for the number of books students read over summer break.



6. Sometimes, distributions are characterized by extreme values that differ greatly from the other observations. These extreme values are called ***outliers***. A data value is an outlier if it is less than $Q_1 - 1.5 \cdot \text{IQR}$ or above $Q_3 + 1.5 \cdot \text{IQR}$.

Example:

This example shows the effect that an outlier can have on a measure of central tendency.

The mean is one of several measures of central tendency that can be used to describe a data set. The main limitation of the mean is that, because every data value directly affects the result, it can be affected greatly by outliers. Consider these two sets of quiz scores:

Student P: {8, 9, 9, 9, 10}
Student Q: {3, 9, 9, 9, 10}

Both students consistently performed well on quizzes and both have the same median and mode score, 9. Student Q, however, has a mean quiz score of 8, while Student P has a mean quiz score of 9. Although many instructors accept the use of a mean as being fair and representative of a student's overall performance in the context of test or quiz scores, it can be misleading because it fails to describe the variation in a student's scores, and the effect of a single score on the mean can be disproportionately large, especially when the number of scores is small.

7. A ***normal distribution*** shows quantitative data that vary randomly from a mean. The pattern follows a symmetrical, bell-shaped curve called a normal curve. As the distance from the mean increases on both sides of the mean, the number of data points decreases. ***Skewness*** refers to the type and degree of a distribution's asymmetry. A distribution is skewed to the left if it has a longer tail on the left side and has a negative value for its skewness. If a distribution has a longer tail on the right, it has positve skewness. Generally distributions have only one peak, but there are distributions called ***bimodal*** or ***multimodal*** where there are two or more peaks, respectively. A distribution can have symmetry but not be a normal distribution. It could be too flat (uniform) or too spindly. A box plot can present a fair representation of a data set's distribution. For a normal distribution, the median should be very close to the middle of the box and the two whiskers should be about the same length.

Important Tip

- The extent to which a data set is distributed normally can be determined by observing its skewness. Most of the data should lie in the middle near the median. The mean and the median should be fairly close. The left and right tails of the distribution curve should taper off. There should be only one peak and it should neither be too high nor too flat.

REVIEW EXAMPLES

1) Josh and Richard each earn tips at their part-time job. This table shows their earnings from tips for five days.

Total Tips by Day

Day	Josh's Tips	Richard's Tips
Monday	$40	$40
Tuesday	$20	$45
Wednesday	$36	$53
Thursday	$28	$41
Friday	$31	$28

a. Who had the greatest median earnings from tips? What is the difference in the median of Josh's earnings from tips and the median of Richard's earnings from tips?

b. What is the difference in the interquartile range for Josh's earnings from tips and Richard's earnings from tips?

Solution:

a. Write Josh's earnings from tips in order from the least to greatest amounts. Then, identify the middle value.

$20, $28, **$31**, $36, $40

Josh's median earnings from tips are $31.

Write Richard's earnings from tips in order from the least to the greatest amounts. Then, identify the middle value.

$28, $40, **$41**, $45, $53

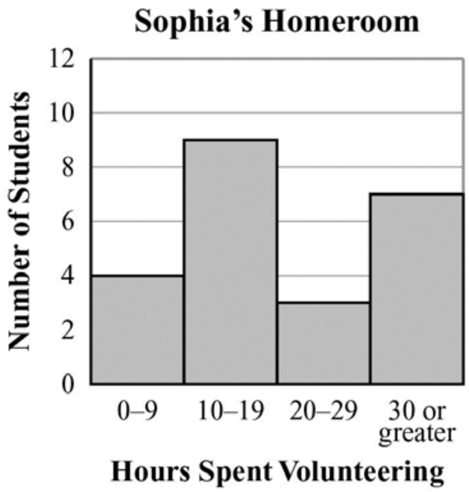
Richard had the greatest median earnings from tips. The difference in the median of the earnings from tips is $41 – $31 = $10.

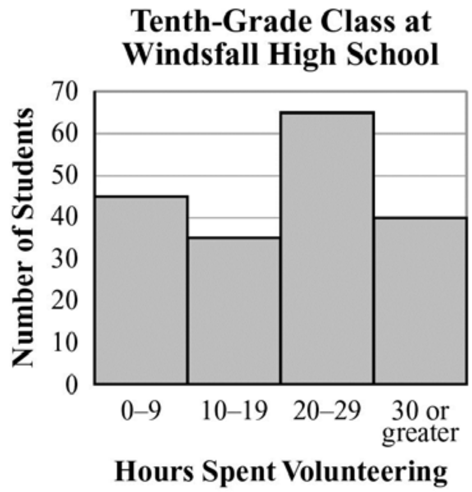
b. For Josh's earnings from tips, the lower quartile is $24 and the upper quartile is $38. The interquartile range $38 – $24, or $14.

For Richard's earnings from tips, the lower quartile is $34 and the upper quartile is $49. The interquartile range $49 – $34, or $15.

The difference in Josh's interquartile range and Richard's interquartile range is $15 – $14, or $1.

2) Sophia is a student at Windsfall High School. These histograms give information about the number of hours spent volunteering by each of the students in Sophia's homeroom and by each of the students in the tenth-grade class at her school.





a. Compare the lower quartiles of the data in the histograms.

b. Compare the upper quartiles of the data in the histograms.

c. Compare the medians of the data in the histograms.

d. Does either histogram reflect a normal distribution? Explain your answer.

Solution:

a. You can add the number of students given by the height of each bar to find that there are 23 students in Sophia's homeroom. The lower quartile is the median of the first half of the data. That would be found within the 10–19 hours interval.

   You can add the numbers of students given by the height of each bar to find that there are 185 students in the tenth-grade class. The lower quartile for this group is found within the 10–19 hours interval.

   The interval of the lower quartile of the number of hours spent volunteering by each student in Sophia's homeroom is the same as the interval of the lower quartile of the number of hours spent volunteering by each student in the tenth-grade class.

b. The upper quartile is the median of the second half of the data. For Sophia's homeroom, that would be found in the 30 or greater interval.

   For the tenth-grade class, the upper quartile is found within the 20–29 hours interval.

   The upper quartile of the number of hours spent volunteering by each student in Sophia's homeroom is more than the upper quartile of the number of hours spent volunteering by each student in the tenth-grade class.

c. The median is the middle data value in a data set when the data values are written in order from least to greatest. The median for Sophia's homeroom is found within the 10–19 hours interval.

The median for the tenth-grade class is found within the 20–29 hours interval.

The median of the number of hours spent volunteering by each student in Sophia's homeroom is less than the number of hours spent volunteering by each student in the tenth-grade class.

d. Neither histogram appears to reflect a normal distribution. If the distribution were normal, most of the number of hours spent volunteering would be represented by the middle bars. The heights of the bars in the histogram for Sophia's homeroom vary without showing a pattern of being skewed. The histogram for the tenth-grade class is slightly skewed to the right.

3) Mr. Storer, the physical education teacher, measured the height of each student in his first period class. He organized his data in this chart.

Height (in inches)	Frequency
42	1
43	2
44	4
45	5
46	4
47	2
48	1

a. Make a dot plot for the data.

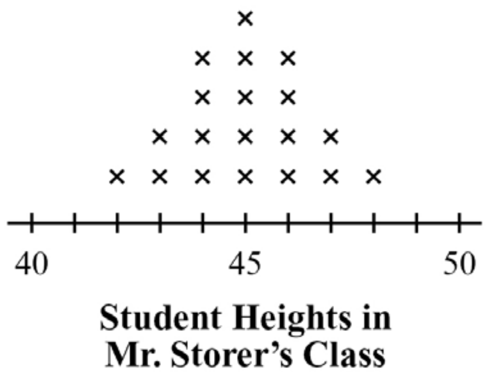
b. Make a histogram for the data.

c. Make a box plot for the data.

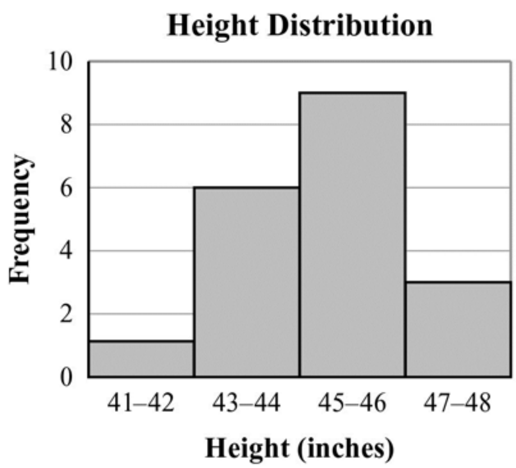
d. Does the distribution of heights appear normal/bell shaped?

Solution:

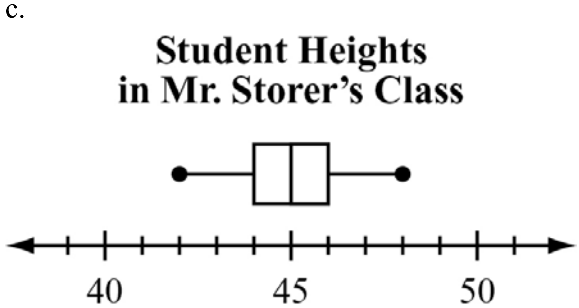
a.



b.

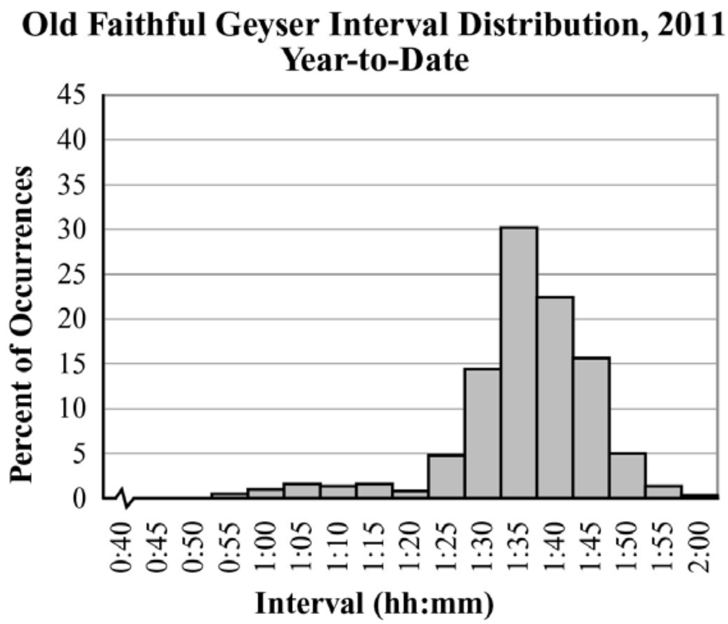


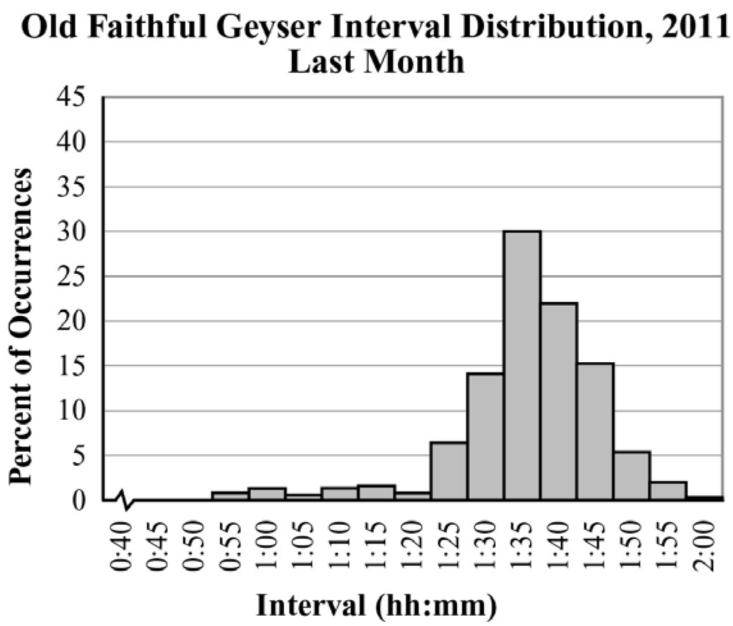
c.

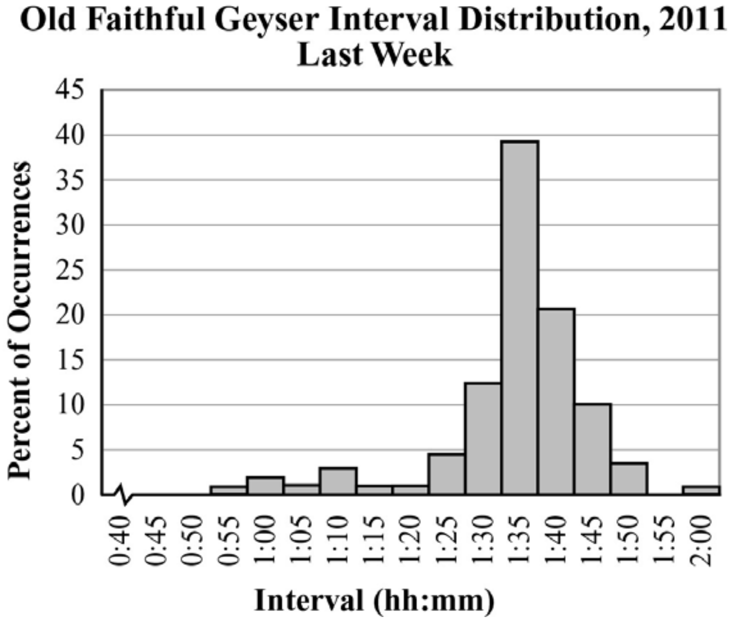


d. Yes, the distribution of student heights appears fairly normal with a concentration in the middle and lesser frequencies in the tails.

4) Old Faithful, a geyser in Yellowstone National park, is renowned for erupting fairly regularly. In more recent times, it has become less predictable. It was observed that the time interval between eruptions was related to the duration of the most recent eruption. The distribution of its interval times for 2011 is shown in the following graphs.







a. Does the Year-to-Date distribution seem normal, skewed, or uniform?

b. Compare Last Week's distribution to Last Month's distribution.

c. What does the Year-to-Date distribution tell you about the interval of time between Old Faithful's eruptions?

Solution:

a. The Year-to-Date distribution appears to be skewed to the left (negative). Most of the intervals approach 90 minutes. In a normal distribution, the peak would be in the middle.

b. Last Week's distribution seems more skewed to the left than Last Month's. It is also more asymmetric. Last Month's distribution appears to have the highest percent of intervals longer than 1 hour 30 minutes between eruptions.

c. The Year-to-Date distribution shows Old Faithful rarely erupts an hour after its previous eruption. Most visitors will have to wait more than 90 minutes to see two eruptions.

EOCT Practice Items

1) This table shows the average low temperature, in ºF, recorded in Macon, GA, and Charlotte, NC, over a six-day period.

Day	1	2	3	4	5	6
Temperature in Macon, GA (in ºF)	71	72	66	69	71	73
Temperature in Charlotte, NC (in ºF)	69	64	68	74	71	75

Which conclusion can be drawn from the data?

A. The interquartile range of the temperatures is the same for both cities.

B. The lower quartile for the temperatures in Macon is lower than the lower quartile for the temperatures in Charlotte.

C. The mean and median temperatures of Macon were higher than the mean and median temperatures of Charlotte.

D. The upper quartile for the temperatures in Charlotte was lower than the upper quartile for the temperatures in Macon.

[Key: C]

2) A school was having a coat drive for a local shelter. A teacher determined the median number of coats collected per class and the interquartile ranges of the number of coats collected per class for the freshmen and for the sophomores.

- **The freshmen collected a median number of coats per class of 10, and the interquartile range was 6.**
- **The sophomores collected a median number of coats per class of 10, and the interquartile range was 4.**

Which range of numbers includes the third quartile of coats collected for both classes?

A. 4 to 14

B. 6 to 14

C. 10 to 16

D. 12 to 15

[Key: C]

3) A reading teacher recorded the number of pages read in an hour by each of her students. The numbers are shown below.

44, 49, 39, 43, 50, 44, 45, 49, 51

For this data, which summary statistic is NOT correct?

A. The minimum is 39.

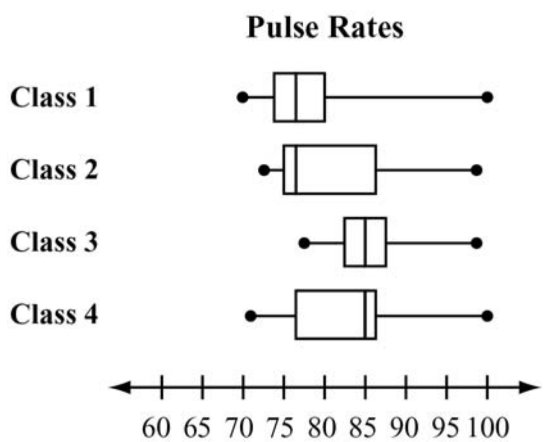
B. The lower quartile is 44.

C. The median is 45.

D. The maximum is 51.

[Key: B]

4) A science teacher recorded the pulse rates for each of the students in her classes after the students had climbed a set of stairs. She displayed the results, by class, using the box plots shown.



Which class generally had the highest pulse rates after climbing the stairs?

A. Class 1
B. Class 2
C. Class 3
D. Class 4

[Key: C]

5) Peter went bowling, Monday to Friday, two weeks in a row. He only bowled one game each time he went. He kept track of his scores below.

Week 1: 70, 70, 70, 73, 75

Week 2: 72, 64, 73, 73, 75

What is the BEST explanation of why Peter's Week 2 mean score was lower than his Week 1 mean score?

A. Peter received the same score three times in Week 1.

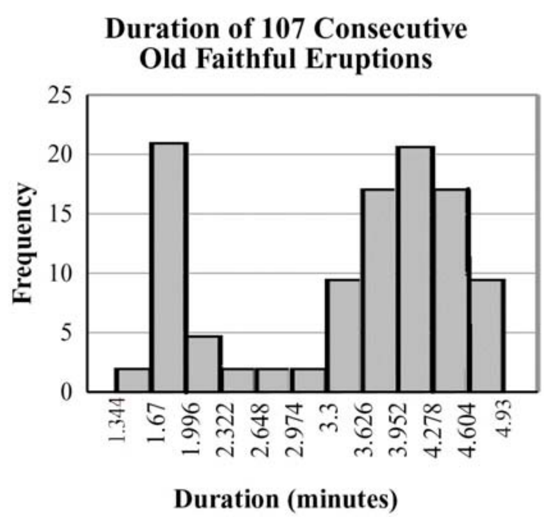
B. Peter had one very low score in Week 2.

C. Peter did not beat his high score from Week 1 in Week 2.

D. Peter had one very high score in Week 1.

[Key: B]

6) This histogram shows the frequency distribution of duration times for 107 consecutive eruptions of the Old Faithful geyser. The duration of an eruption is the length of time, in minutes, from the beginning of the spewing of water until it stops. What is the BEST description for the distribution?



A. bimodal

B. uniform

C. multiple outlier

D. skewed to the right

[Key: A]

SUMMARIZE, REPRESENT, AND INTERPRET DATA ON TWO CATEGORICAL AND QUANTITATIVE VARIABLES

KEY IDEAS

1. There are essentially two types of data: ***categorical*** and ***quantitative***. Examples of categorical data are: color, type of pet, gender, ethnic group, religious affiliation, etc. Examples of quantitative data are: age, years of schooling, height, weight, test score, etc. Researchers use both types of data but in different ways. Bar graphs and pie charts are frequently associated with categorical data. Box plots, dot plots, and histograms are used with quantitative data. The measures of central tendency (mean, median, and mode) apply to quantitative data. Frequencies can apply to both categorical and quantitative.

2. ***Bivariate data*** consists of pairs of linked numerical observations, or frequencies of things in categories. Numerical bivariate data can be presented as ordered pairs and in any way that ordered pairs can be presented: as a set of ordered pairs, as a table of values, or as a graph on the coordinate plane.

   Categorical example: frequencies of gender and club memberships for 9th graders.

   A bivariate or ***two-way frequency chart*** is often used with data from two categories. Each category is considered a variable, and the categories serve as labels in the chart. Two-way frequency charts are made of cells. The number in each cell is the frequency of things that fit both the row and column categories for the cell. From the two-way frequency chart below, we see that there are 12 males in the band and 3 females in the chess club.

Participation in School Activities			
School Club	**Gender**		
	Male	Female	**Totals**
Band	12	21	33
Chorus	15	17	32
Chess	16	3	19
Latin	7	9	16
Yearbook	28	7	35
Totals	78	57	**135**

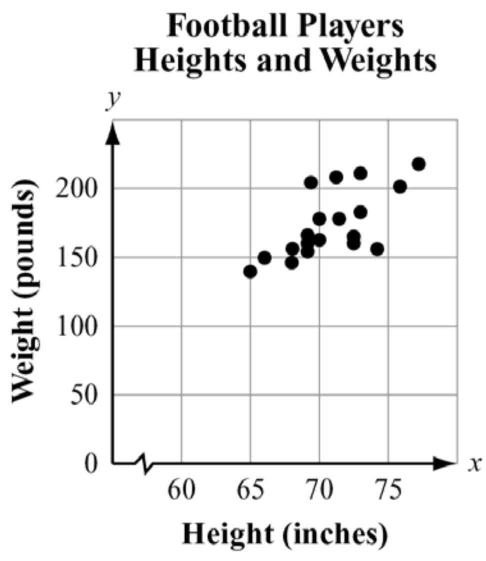
If no person or thing can be in more than one category per scale, the entries in each cell are called ***joint frequencies***. The frequencies in the cells and the totals tell us about the percentages of students engaged in different activities based on gender. For example, we can determine from the chart that if we picked at random from the students, we are least likely to find a female in the chess club because only 3 of 135 students are females in the chess club. The most popular club is yearbook, with 35 of 135 students in that club. The values in the table can be converted to percents, which will give us an idea of the composition of each club by gender. We see that close to 14% of the students are in the chess club, and there are more than five times as many boys as girls.

Participation in School Activities			
School Club	**Gender**		
	Male	**Female**	**Totals**
Band	8.9%	15.5%	24.4%
Chorus	11.1%	12.6%	23.7%
Chess	11.9%	2.2%	14.1%
Latin	5.2%	6.7%	11.9%
Yearbook	20.7%	5.2%	25.9%
Totals	57.8%	42.2%	**100%**

There are also what we call ***marginal frequencies*** in the bottom and right margins (the shaded cells in the table above). These frequencies lack one of the categories. For our example, the frequencies at the bottom represent percents of males and females in the school population. The marginal frequencies on the right represent percents of club membership.

Lastly, associated with two-way frequency charts are ***conditional frequencies***. These are not usually in the body of the chart, but can be readily calculated from the cell contents. One conditional frequency would be the percent of chorus members that are female. The working condition is that the person is female. If 12.6% of the entire school population is females in the chorus, and 42.3% of the student body is female, then 12.6% / 42.3%, or 29.8%, of the females in the school are in the chorus (also 17 of 57 girls).

Quantitative example: Consider this chart of heights and weights of players on a football team.



A scatter plot is often used to present bivariate quantitative data. Each variable is represented on an axis and the axes are labeled accordingly. Each point represents a player's height and weight. For example, one of the points represents a height of 66 inches and weight of 150 pounds. The scatter plot shows two players standing 70 inches tall because there are two dots on that height.

3. A ***scatter plot*** displays data as points on a grid using the associated numbers as coordinates. The way the points are arranged by themselves in a scatter plot may or may not suggest a relationship between the two variables. In the scatter plot about the football players shown above, it appears there may be a relationship between height and weight because, as the players get taller, they seem to generally increase in weight; that is, the points are positioned higher as you move to the right. Bivariate data may have an underlying relationship that can be modeled by a mathematical function. For the purposes of this unit we will consider linear models.

Example:

Melissa would like to determine whether there is a relationship between study time and mean test scores. She recorded the mean study time per test and the mean test score for students in three different classes.

This is the data for Class 1.

Class 1 Test Score Analysis	
Mean Study Time (hours)	**Mean Test Score**
0.5	63
1	67
1.5	72
2	76
2.5	80
3	85
3.5	89

Notice that, for these data, as the mean study time increases, the mean test score increases. It is important to consider the ***rate of increase*** when deciding which algebraic model to use. In this case, the mean test score increases by approximately 4 points for each 0.5-hour increase in mean study time. When the rate of increase is close to constant, as it is here, the best model is most likely a linear function.

This next table shows Melissa's data for Class 2.

Class 2 Test Score Analysis	
Mean Study Time (hours)	**Mean Test Score**
0.5	60
1	61
1.5	63
2	68
2.5	74
3	82
3.5	93

In these data as well, the mean test score increases as the mean study time increases. However, the rate of increase is not constant. The differences between each successive mean test score are 1, 2, 5, 6, 8, and 11. The ***second differences*** are 1, 3, 1, 2, and 3. Since the second differences are fairly close to constant, it is likely that a different model such as an exponential function would be employed for the Class 2 data.

This table shows Melissa's data for Class 3.

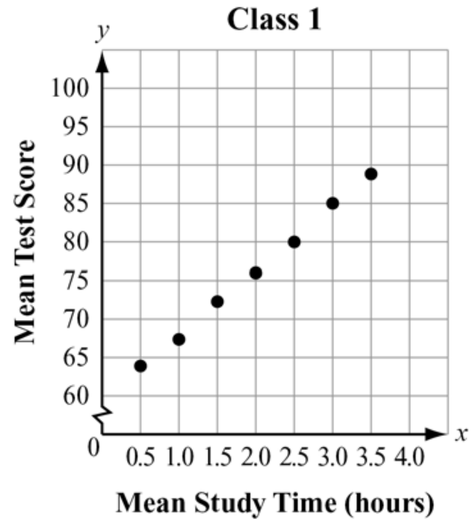
Class 3 Test Score Analysis	
Mean Study Time (hours)	**Mean Test Score**
0.5	71
1	94
1.5	87
2	98
2.5	69
3	78
3.5	91

In these data, as the mean study time increases, there is no consistent pattern in the mean test score. As a result, there does not appear to be any clear relationship between the mean study time and mean test score for this particular class.

Often, patterns in bivariate data are more easily seen when the data is plotted on a coordinate grid.

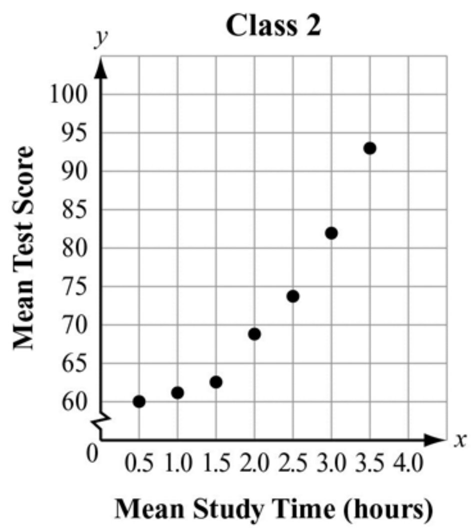
Example:

This graph shows Melissa's data for Class 1.



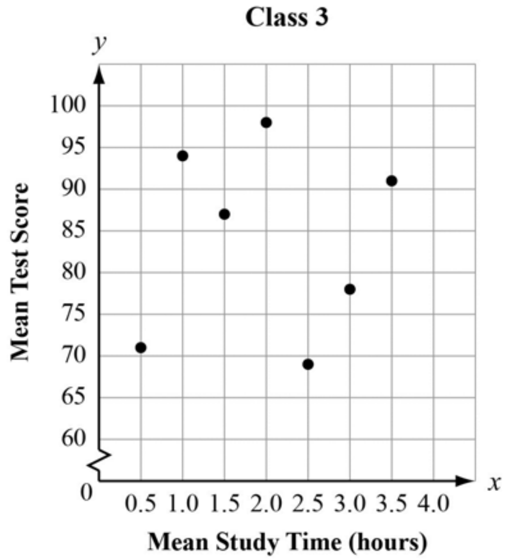
In this graph, the data points are all very close to being on the same line. This is further confirmation that a linear model is appropriate for this class.

This graph shows Melissa's data for Class 2.



In this graph, the data points appear to lie on a curve, rather than on a line, with a rate of increase that increases as the value of x increases. It appears that an exponential model may be more appropriate than a linear model for these data.

This graph shows Melissa's data for Class 3.



In this graph, the data points do not appear to lie on a line or on a curve. Neither a linear model nor an exponential model is appropriate to represent the data.

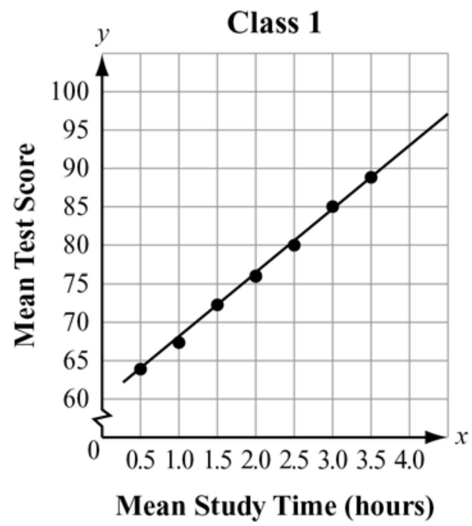
4. A ***line of best fit*** (trend or regression line) is a straight line that best represents the data on a scatter plot. This line may pass through some of the points, none of the points, or all of the points. In the previous examples, only the Class 1 scatter plot looks like a linear model would be a good fit for the points. In the other classes, a curved graph would seem to pass through more of the points. For Class 2, perhaps an exponential model would produce a better fitting curved.

When a linear model is indicated there are several ways to find a function that approximates the y-value for any given x-value. A method called ***regression*** is the best way to find a line of best fit, but it requires extensive computations and is generally done on a computer or graphing calculator.

Items on the EOCT ask students to determine the equation of a line of best fit when given a graph. Students may also be asked to estimate a line of best fit for a given scatter plot. Items may require data interpretation.

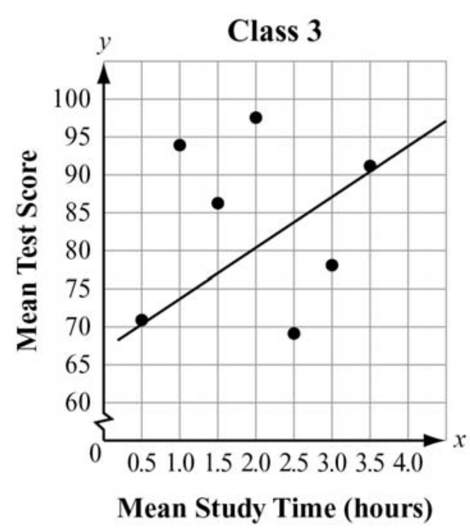
Example:

This graph shows Melissa's data for Class 1 with the line of best fit added. The equation of the line is $y = 8.8x + 58.4$.



Notice that five of the seven data points are on the line. This represents a very strong positive relationship for study time and test scores, since the line of best fit is positive and a very tight fit to the data points.

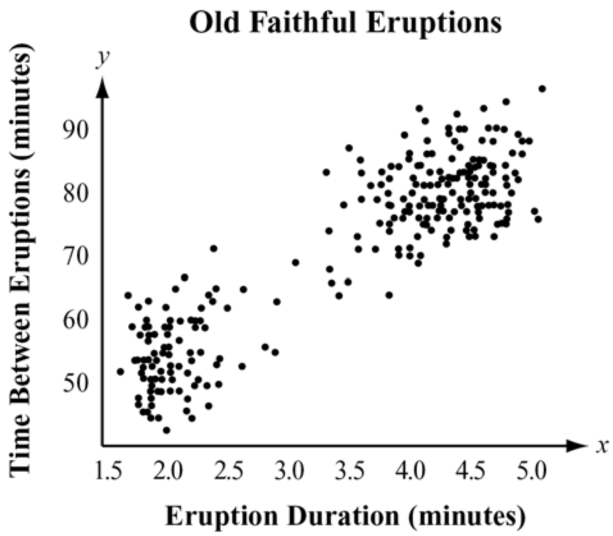
This next graph shows Melissa's data for Class 3 with the line of best fit added. The equation of the line is $y = 0.8\,x + 83.1$.



Although a line of best fit can be calculated for this set of data, notice that most of the data points are not very close to the line. In this case, although there is some correlation between study time and test scores, the amount of correlation is very small.

REVIEW EXAMPLES

1) Barbara is considering visiting Yellowstone National Park. She has heard about Old Faithful, the geyser, and she wants to make sure she sees it erupt. At one time it erupted just about every hour. That is not the case today. The time between eruptions varies. Barbara went on the Web and found a scatter plot of how long an eruption lasted compared to the wait time between eruptions. She learned that, in general, the longer the wait time, the longer the eruption lasts. The eruptions take place anywhere from 45 minutes to 125 minutes apart. They currently average 90 minutes apart.



a. For an eruption that lasts 4 minutes, about how long would the wait time be for the next eruption?

b. What is the shortest duration time for an eruption?

c. Do you think the scatter plot could be modeled with a linear function?

Solution:

a. After a 4-minute eruption, it would be between 75 to 80 minutes for the next eruption.

b. The shortest eruptions appear to be a little more than 1.5 minutes (1 minute and 35 seconds.)

c. There seem to be two major regions in the scatter plot, so a single line may not be a good predictor of both regions.

2) The environment club is interested in the relationship between the number of canned beverages sold in the cafeteria and the number of cans that are recycled. The data they collected are listed in this chart.

Beverage Can Recycling								
Number of Canned Beverages Sold	18	15	19	8	10	13	9	14
Number of Cans Recycled	8	6	10	6	3	7	5	4

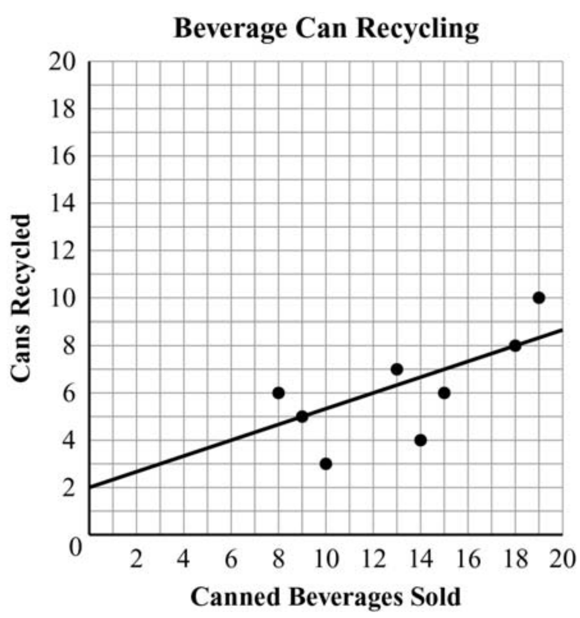
Find an equation of a line of best fit for the data.

Solution:

Write the data as ordered pairs.

$$\{(8, 6)\ (9, 5)\ (10, 3)\ (13, 7)\ (14, 4)\ (15, 6)\ (18, 8)\ (19, 10)\}$$

Plot the ordered pairs on a coordinate grid and draw a line that approximates the trend of the data. Draw the line so that it has about the same number of points above and below the line. It does not always have to cross directly through data points.



To find the slope of the line, choose two points on the line, such as (9, 5) and (18, 8). Then calculate the slope.

$$\text{Slope} = \frac{8-5}{18-9} = \frac{3}{9} = \frac{1}{3}$$

The line appears to cross the y-axis at (0, 2), so estimate the y-intercept of this line as 2.

So, the equation of a line of best fit for this data could be $y = \frac{1}{3}x + 2$.

2) A fast food restaurant wants to determine if the season of the year affects the choice of soft-drink size purchased. They surveyed 278 customers and the table below shows their results. The drink sizes were small, medium, large, and jumbo. The seasons of the year were spring, summer, and fall. In the body of the table, the cells list the number of customers that fit both row and column titles. On the bottom and in the right margin are the totals.

	Spring	**Summer**	**Fall**	**TOTALS**
Small	24	22	18	64
Medium	23	28	19	70
Large	18	27	29	74
Jumbo	16	21	33	70
TOTALS	81	98	99	278

a. In which season did the most customers prefer jumbo drinks?

b. What percent of those surveyed purchased the small drinks?

c. What percent of those surveyed purchased medium drinks in the summer?

d. What do you think the fast-food restaurant learned from their survey?

Solution:

a. The most customers preferred jumbo drinks in the fall.

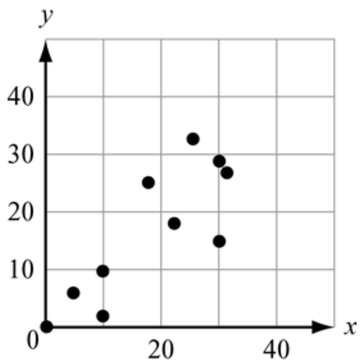
b. Twenty-three percent (64/278 = 23%) of the 278 surveyed purchased the small drinks.

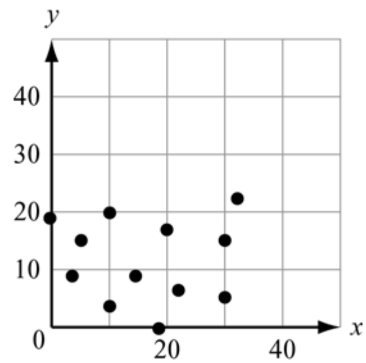
c. Ten percent (28/278 = 10%) of those customers surveyed purchased medium drinks in the summer.

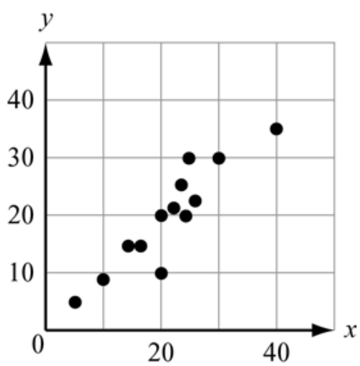
d. The fast food restaurant probably learned that customers tend to purchase the larger drinks in the fall, the smaller drinks in the spring and summer.

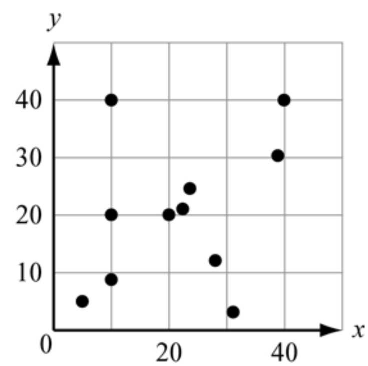
EOCT Practice Items

1) Which graph MOST clearly displays a set of data for which a linear function is the model of best fit?

A.


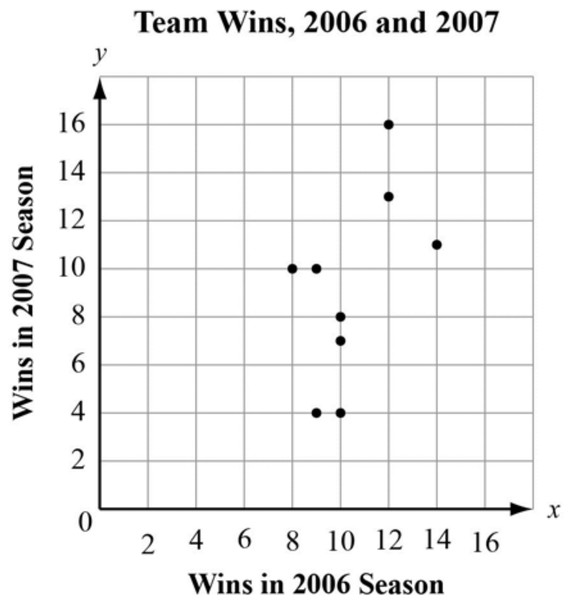
B.


C.


D.


[Key: C]

2) This graph plots the number of wins in the 2006 and 2007 seasons for a sample of professional football teams.



Which equation BEST represents a line that matches the trend of this data?

A. $y = \frac{1}{2}x$

B. $y = \frac{1}{2}x + 8$

C. $y = 2x - 6$

D. $y = 2x - 12$

[Key: D]

INTERPRET LINEAR MODELS

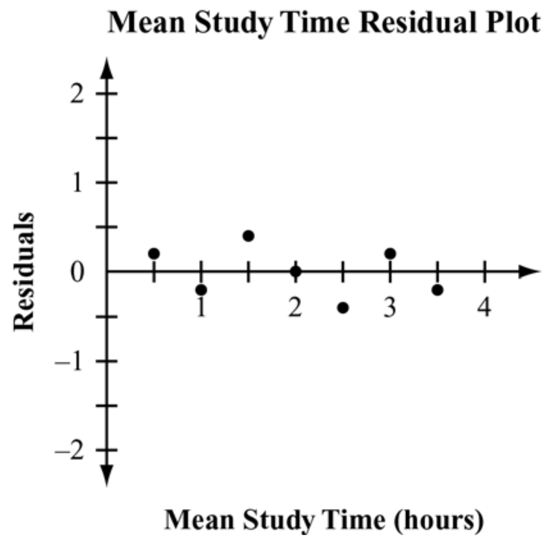
KEY IDEAS

1. Once a model for the scatter plot is determined, its goodness of fit is very important. The goodness of fit depends on the model's accuracy in predicting values. ***Residuals***, or error distances, are used to measure the goodness of fit. A residual is the difference between the observed value and the model's predicted value. For a regression model, a residual = observed value – predicted value. A residual plot is a graph that shows the residual values on the vertical axis and the independent variable (x) on the horizontal axis. A residual plot shows where the model fits best, and where the fit is worst. A good regression fit has very short residuals.

Example:

Take the data from the test scores for Class 1 used in the last section. The observed mean test scores were 63, 67, 72, etc. The best fit model was a linear model with the equation $y = 8.8x + 58.4$. We can calculate the residuals for this data and consider the fit of the regression line.

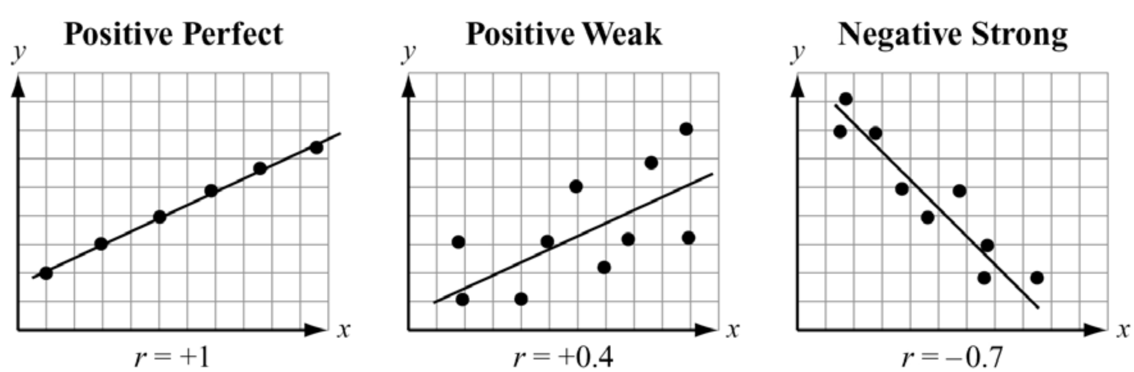
Mean Study Time (hours)	Mean Test Score	Predicted Score $y = 8.8x + 58.4$	Residual
0.5	63	62.8	0.2
1.0	67	67.2	–0.2
1.5	72	71.6	0.4
2.0	76	76	0
2.5	80	80.4	–0.4
3.0	85	84.8	0.2
3.5	89	89.2	–0.2



Notice the numbers in the residual column tell us how far the predicted mean test score was from the observed, as seen in the regression scatter plot for Class 1. The regression passes through one of the actual points in the plot of the points where the residual is 0. Notice also, the residuals add up to 0. Residuals add up to 0 for a properly calculated regression line. The goal is to minimize all of the residuals.

2. A ***correlation coefficient*** is a measure of the strength of the linear relationship between two variables. It also indicates whether the dependent variable, *y*, grows along with *x*, or *y* get smaller as *x* increases. The correlation coefficient is a number between –1 and + 1 including –1 and +1. The letter *r* is usually used for the correlation coefficient. When the correlation is positive, the line of best fit will have positive slope and both variables are growing. However, if the correlation coefficient is negative, the line of best fit has negative slope and the dependent variable is decreasing. The numerical value is an indicator of how closely the data points are modeled by a linear function.

Examples:



The correlation between two variables is related to the slope and the goodness of the fit of a regression line. However, data in scatter plots can have the same regression lines and very different correlations. The correlation's sign will be the same as the slope of the regression line. The correlation's value depends on the dispersion of the data points and their proximity to the line of best fit.

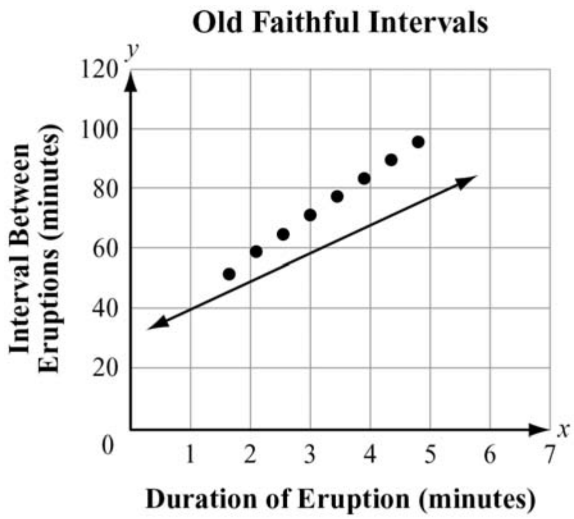
Example:

Earlier we saw that the interval between eruptions of Old Faithful is related to the duration of the most recent eruption. Years ago, the National Park Service had a simple linear equation they used to help visitors determine when the next eruption would take place. Visitors were told to multiply the duration of the last eruption by 10 and add 30 minutes (I = 10 • D + 30). We can look at a 2011 set of data for Old Faithful, with eight data points, and see how well that line fits today. The data points are from a histogram with intervals of 0.5 minutes for x-values. The y-values are the average interval time for an eruption in that duration interval.

Old Faithful Eruptions

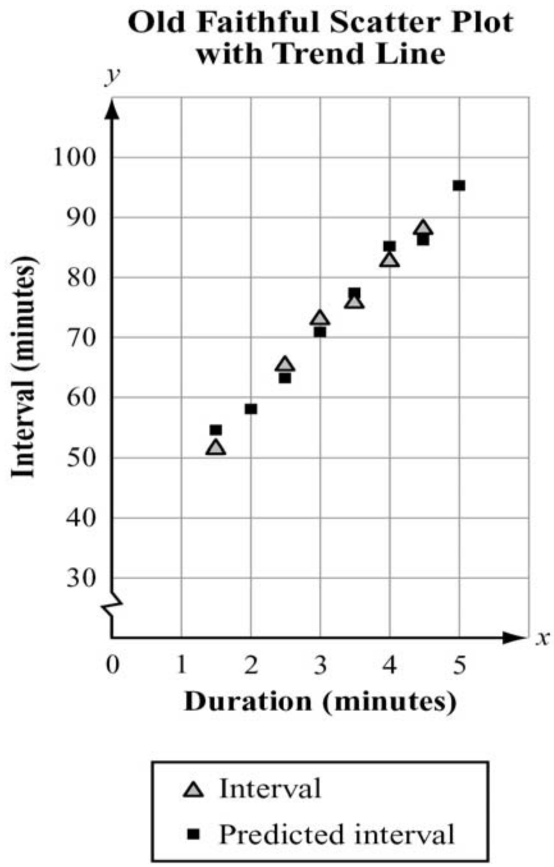
Duration (x)	Interval (y)	Prediction	Residual
1.50	51.00	45	6
2.00	58.00	50	8
2.50	65.00	55	10
3.00	71.00	60	11
3.50	76.00	65	11
4.00	82.00	70	12
4.50	89.00	75	14
5.00	95.00	80	15

The residuals display a clear pattern. The Park Service's regression line on the scatter plot shows the same reality. They keep increasing by small increments. The formula, $I = 10 \cdot D + 30$ no longer works as a good predictor. In fact, it is a worse predictor for longer eruptions.



Instead of using the old formula, the National Park Service has a chart like the one in this example for visitors when they want to gauge how long it will be until the next eruption. We can take the chart the National Park Service uses and see what the new regression line would be. But first, does the scatter plot above look like we should use a linear model? And, do the *y*-values of the data points in the chart have roughly a constant difference?

The answer to both questions is “yes.” The data points do look as though a linear model would fit. The differences in intervals are all 5s, 6s, and 7s. In cases like this, you can use technology to find a linear regression equation.



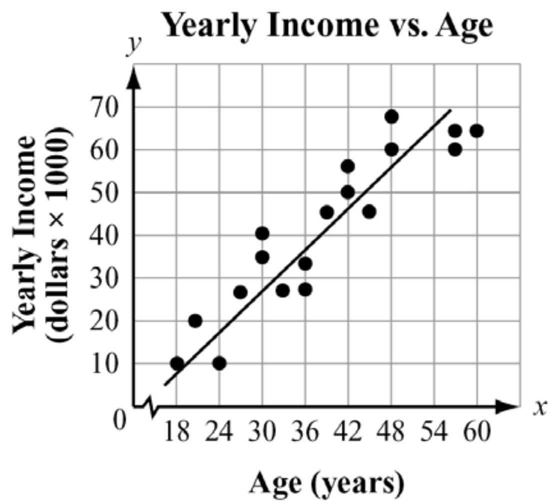
The technology determines data points for the new trend line that appear to fit the observed data points much better than the old line. The interval predicting equation has new parameters for the model, $a = 12.36$ (up 2.36 minutes) and $b = 33.2$ (up 3.2 minutes). The new regression line would be $y = 12.36\,x + 33.2$. While the new regression line appears to come much closer to the observed data points, there are still residuals, especially for lesser duration times. The scientists at Yellowstone Park believe that there probably should be two regression lines now: one for use with shorter eruptions and another for longer eruptions. As we saw from the frequency distribution earlier, Old Faithful currently tends to have longer eruptions, which are farther apart.

The technology also provides a correlation coefficient. From the picture of the regression points above, it looks like the number should be positive and fairly close to 1. The technology provided a correlation coefficient of 0.9992. Indeed, the length of the interval between Old Faithful's eruptions is very strongly related to its most recent eruption duration. The direction is positive, confirming the longer the eruption, the longer the interval between eruptions.

It is very important to point out that the length of Old Faithful's eruptions does not directly cause the interval to be longer or shorter between eruptions. The reason it takes longer for Old Faithful to erupt again after a long eruption is not technically known. However, with a correlation coefficient so close to 1, the two variables are closely related to one another. However, you should never confuse correlation with causation. For example, research shows a correlation between income and age, but aging is not the reason for an increased income. Not all people earn more money the longer they live. Variables can be related to each other without one causing the other.

REVIEW EXAMPLES

1) This scatter plot suggests a relationship between the variables age and income.

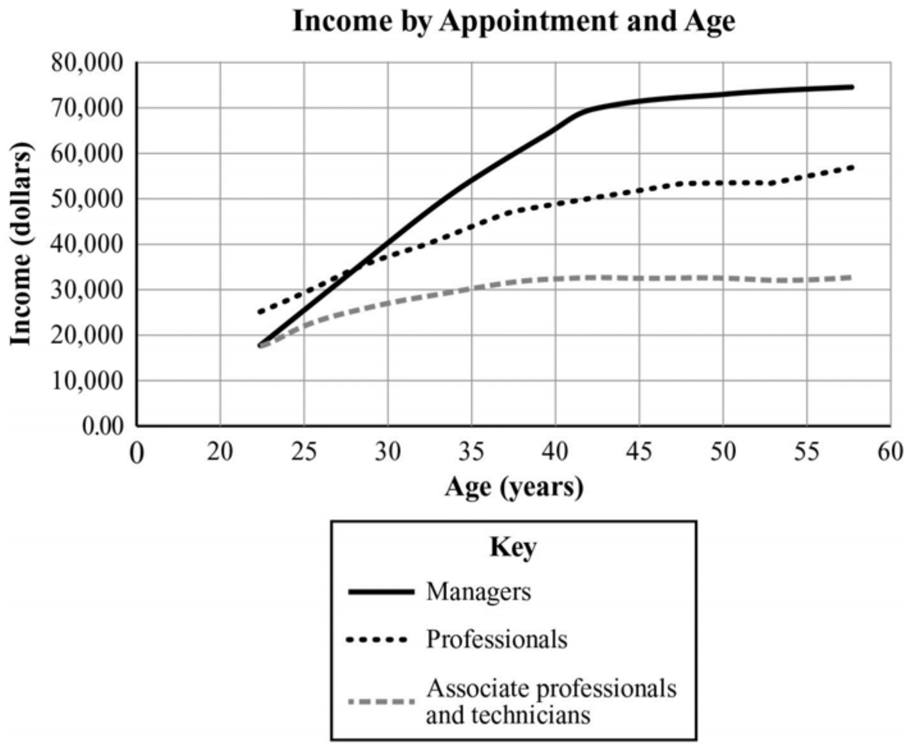


a. What type of a relationship is suggested by the scatter plot (positive/negative, weak/strong)?
b. What is the domain of ages considered by the researchers?
c. What is the range of incomes?
d. Do you think age causes income level to increase? Why or why not?

Solution:

a. The scatter plot suggests a fairly strong positive relationship between age and yearly income.
b. The domain of ages considered is 18 to 60 years.
c. The range of incomes appears to be \$10,000 to \$70,000.
d. No; the variables are related, but age does not cause income to increase.

2) A group of researchers looked at income and age in Singapore. Their results are shown below. They used line graphs instead of scatter plots so they could consider the type of occupation of the wage earner.

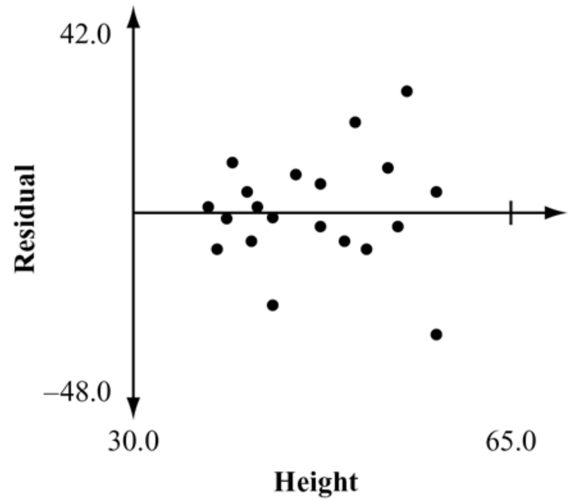


a. Does there appear to be a relationship between age and income?
b. Do all three types of employees appear to share the same benefit of aging when it comes to income?
c. Does a linear model appear to fit the data for any of the employee types?
d. Does the relationship between age and income vary over a person's lifetime?

Solution:

a. Yes, as people get older their income tends to increase.
b. No. The incomes grow at different rates until age 40. For example, the managers' incomes grow faster than the other employee types until age 40.
c. No. The rate of growth appears to vary for all three categories, making a linear model unsuitable for modeling this relationship over a longer domain.
d. Yes, after about age 40, the income for each type of employee grows slower than it did from age 20 to 40.

3) Consider the residual plot below. Each vertical segment represents the difference between an observed weight and a predicted weight of a person, based on height.



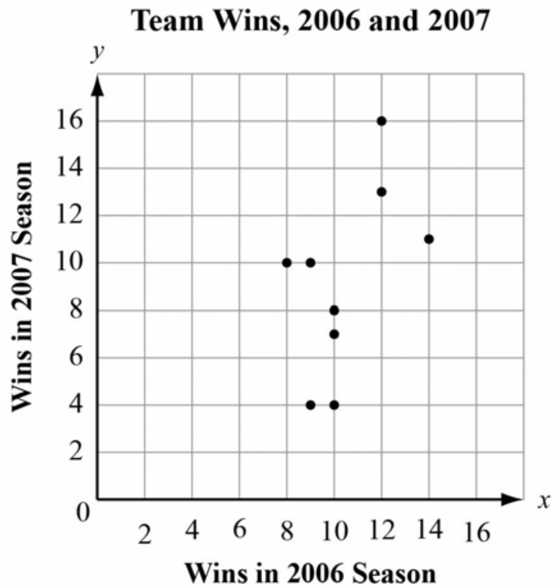
a. Do you think the regression line is a good predictor of weight?
b. Why do the residuals appear to be getting longer for greater heights?

Solution:

a. The regression line does not appear to be a good predictor. Some of the predicted weights were off. The points on the regression plot are dispersed.
b. There are many other factors not shown in the data that can affect weight for taller people.

EOCT Practice Items

1) This graph plots the number of wins in the 2006 and 2007 seasons for a sample of professional football teams.

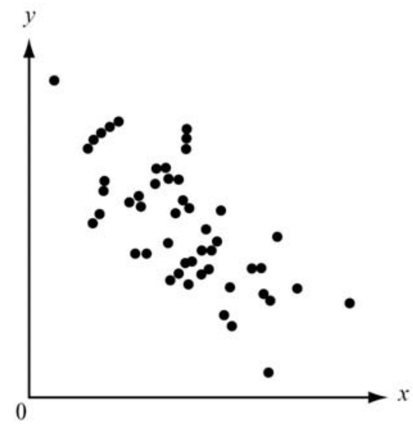


Based on the regression model, what is the predicted number of 2007 wins for a team that won 5 games in 2006?

A. 0
B. 3
C. 8
D. 10

[Key: A]

2) Which BEST describes the correlation of the two variables shown in the scatter plot?



A. weak positive
B. strong positive
C. weak negative
D. strong negative

[Key: D]

Unit 5: Transformations in the Coordinate Plane

In this unit, students review the definitions of three types of transformations that preserve distance and angle: rotations, reflections, and translations. They investigate how these transformations are applied in the coordinate plane as functions, mapping pre-image points (inputs) to image points (outputs). Using their knowledge of basic geometric figures and special polygons, they apply these transformations to obtain images of given figures. They also specify transformations that can be applied to obtain a given image from a given pre-image, including cases in which the image and pre-image are the same figure.

KEY STANDARDS

Experiment with transformations in the plane

MCC9-12.G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

MCC9-12.G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

MCC9-12.G.CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

MCC9-12.G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

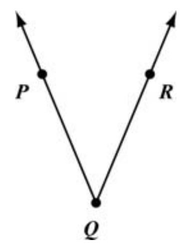
MCC9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

EXPERIMENT WITH TRANSFORMATIONS IN THE PLANE

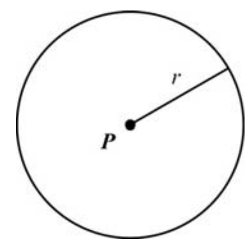
KEY IDEAS

1. A ***line segment*** is part of a line; it consists of two points and all points between them. An ***angle*** is formed by two rays with a common endpoint. A ***circle*** is the set of all points in a plane that are a fixed distance from a given point, called the center; the fixed distance is the ***radius***. ***Parallel lines*** are lines in the same plane that do not intersect. ***Perpendicular lines*** are two lines that intersect to form right angles.

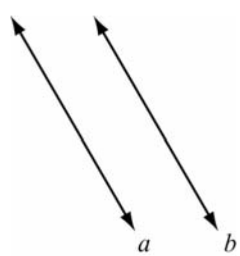
Line segment *AB*



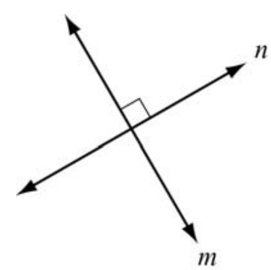
Angle *PQR*



Circle *P*
with radius *r*

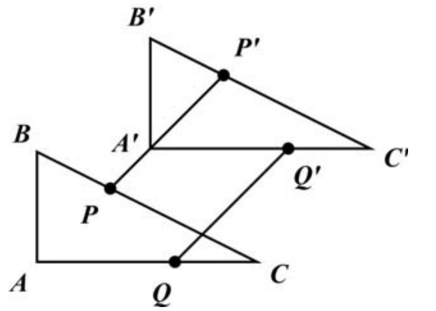


Parallel lines *a* and *b*



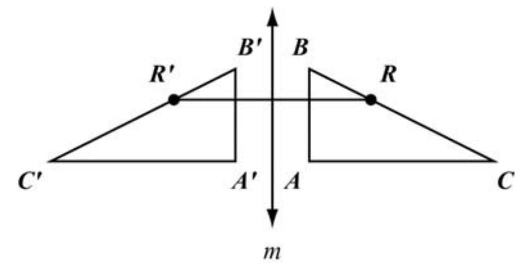
Perpendicular lines *m* and *n*

2. A ***transformation*** is an operation that maps, or moves, a pre-image onto an image. In each transformation defined below, it is assumed that all points and figures are in one plane. In each case, $\triangle ABC$ is the pre-image and $\triangle A'B'C'$ is the image.



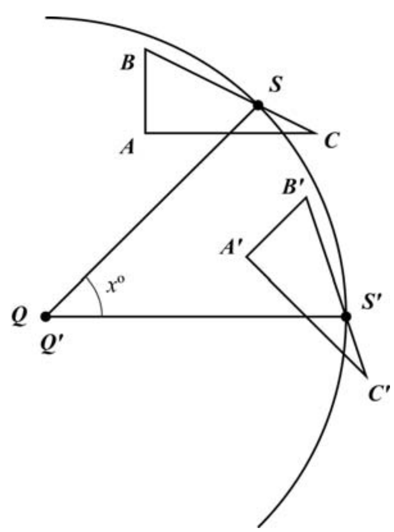
A ***translation*** maps every two points P and Q to points P' and Q' so that the following properties are true:

- $PP' = QQ'$
- $\overline{PP'} \parallel \overline{QQ'}$



A ***reflection*** across a line m maps every point R to R' so that the following properties are true:

- If R is not on m, then m is the perpendicular bisector of $\overline{RR'}$.
- If R is on m, then R and R' are the same point.

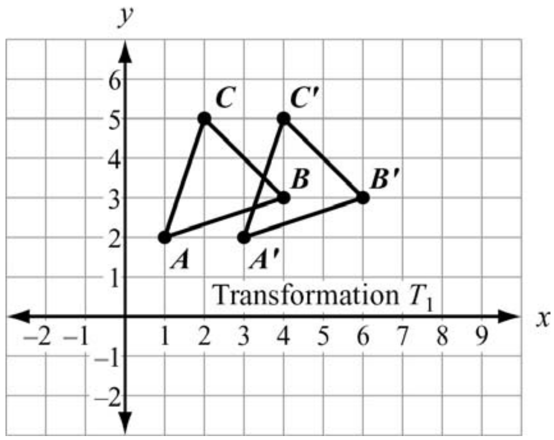


A ***rotation*** of $x°$ about a point Q maps every point S to S' so that the following properties are true:

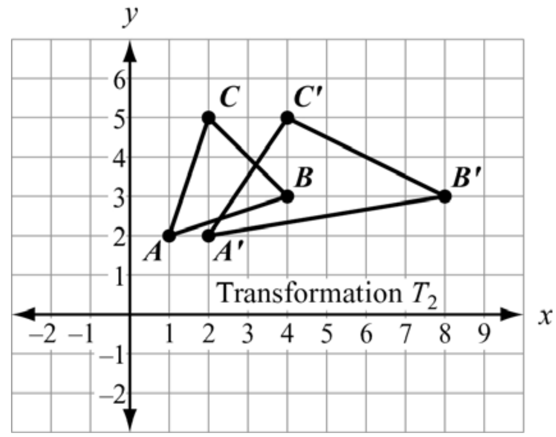
- $SQ = S'Q$ and $m\angle SQS' = x°$.
- Pre-image point Q and image point Q' are the same.

Note: $\overline{QS}$ and $\overline{QS'}$ are radii of a circle with center Q.

3. A transformation in a coordinate plane can be described as a function that maps pre-image points (inputs) to image points (outputs). Translations, reflections, and rotations all preserve distance and angle measure because, for each of those transformations, the pre-image and image are congruent. But some types of transformations do not preserve distance and angle measure because the pre-image and image are not congruent.



T_1: $(x, y) \rightarrow (x + 2, y)$
T_1 translates $\triangle ABC$ to the right 2 units.
T_1 preserves distance and angle measure because $\triangle ABC \cong \triangle A'B'C'$.



T_2: $(x, y) \rightarrow (2x, y)$
T_2 stretches $\triangle ABC$ horizontally by the factor 2.
T_2 preserves neither distance nor angle measure.

4. If vertices are not named, then there might be more than one transformation that will accomplish a specified mapping. If vertices are named, then they must be mapped in a way that corresponds to the order in which they are named.

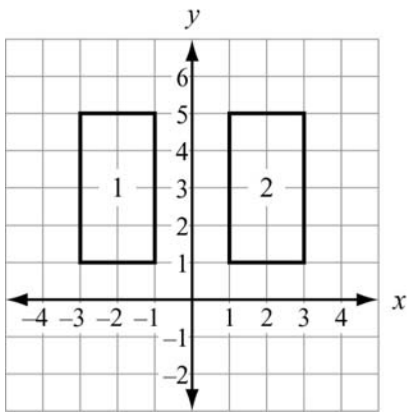
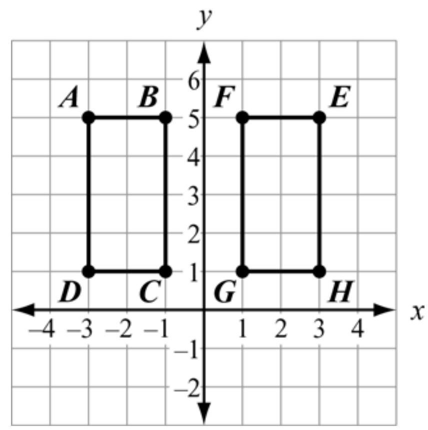


Figure 1 can be mapped to Figure 2 by either of these transformations:

- a reflection across the y-axis (the upper-left vertex in Figure 1 is mapped to the upper-right vertex in Figure 2), or
- a translation 4 units to the right (the upper-left vertex in Figure 1 is mapped to the upper-left vertex in Figure 2.).

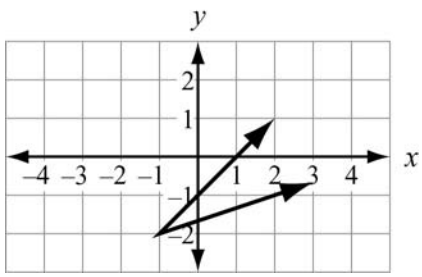


$ABCD$ can be mapped to $EFGH$ by a reflection across the y-axis, but not by a translation.
The mapping of $ABCD \rightarrow EFGH$ requires these vertex mappings:
$A \rightarrow E$, $B \rightarrow F$, $C \rightarrow G$, and $D \rightarrow H$.

REVIEW EXAMPLES

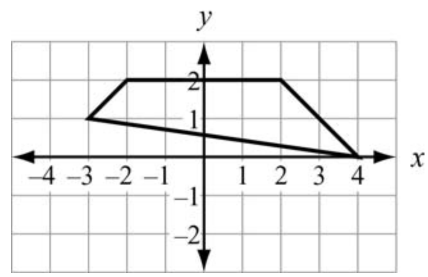
1) Draw the image of each figure, using the given transformation.

a.



Use the translation $(x, y) \rightarrow (x - 3, y + 1)$.

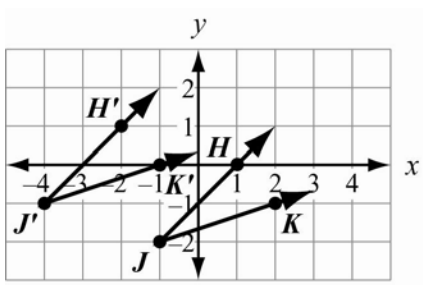
b.



Reflect across the x-axis.

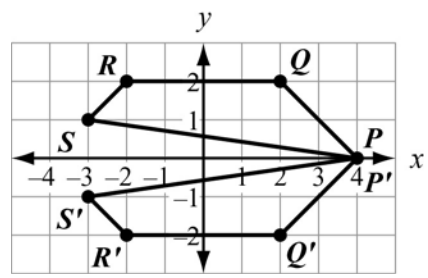
Solution:

a.



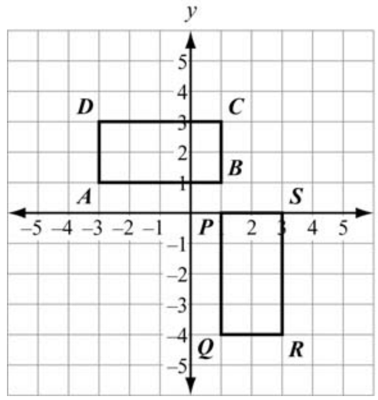
Identify the vertex and a point on each side of the angle. Translate each point 3 units left and 1 unit up. The image of given $\angle HJK$ is $\angle H'J'K'$.

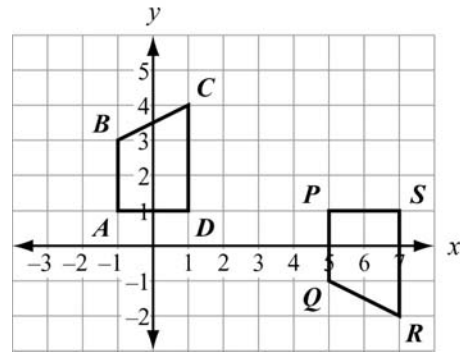
b.



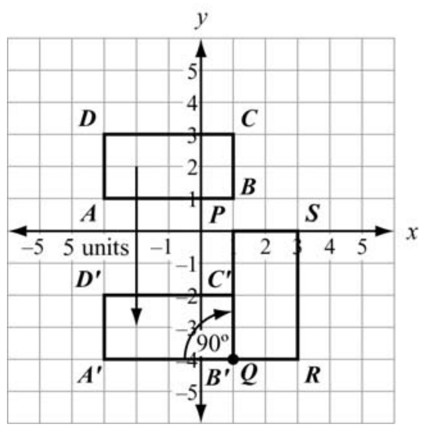
Identify the vertices. The reflection image of each point (x, y) across the x-axis is $(x, -y)$.
The image of given polygon $PQRS$ is $P'Q'R'S'$, where P and P' are the same.

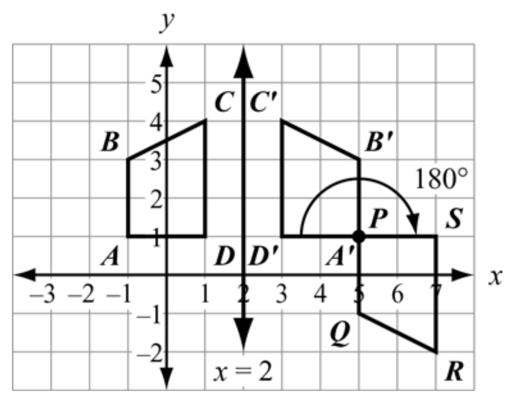
2) Specify a sequence of transformations that will map *ABCD* to *PQRS* in each case.

a. 

b. 

Solution:

a. 

b. 

Translate *ABCD* down 5 units to obtain $A'B'C'D'$. Then rotate $A'B'C'D'$ clockwise 90° about point *B′* to obtain *PQRS*.

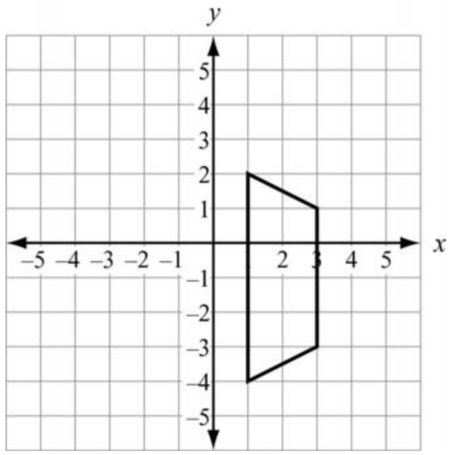
Reflect *ABCD* across the line $x = 2$ to obtain $A'B'C'D'$. Then rotate $A'B'C'D'$ 180° about point *A′* to obtain *PQRS*.
Note that *A′* and *P* are the same point.

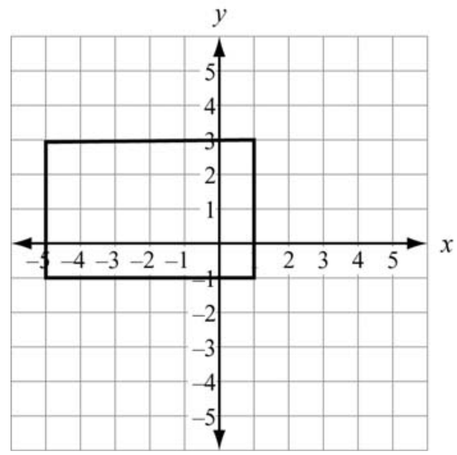
Note that there are other sequences of transformations that will also work for each case.

Important Tip

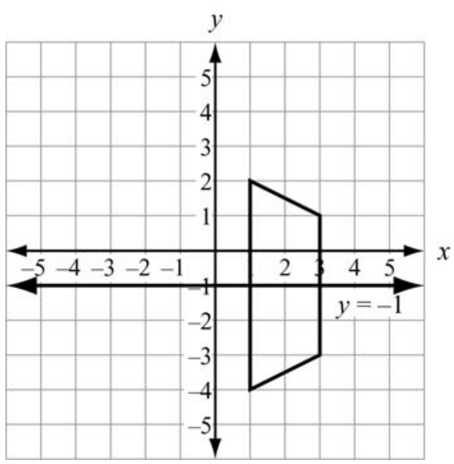
- A 180° rotation clockwise is equivalent to a 180° rotation counterclockwise.

3) Describe every transformation that maps each given figure to itself.

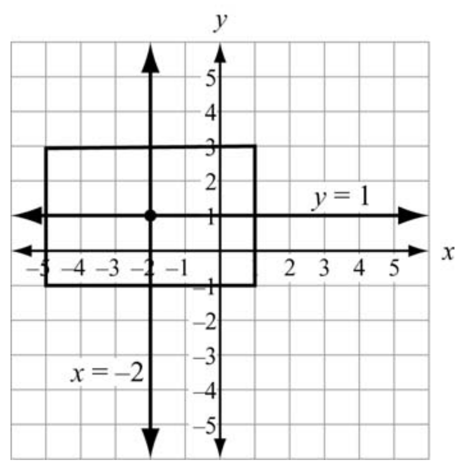
a. 

b. 

Solution:

a. 

There is only one transformation: Reflect the figure across the line $y = -1$.

b. 

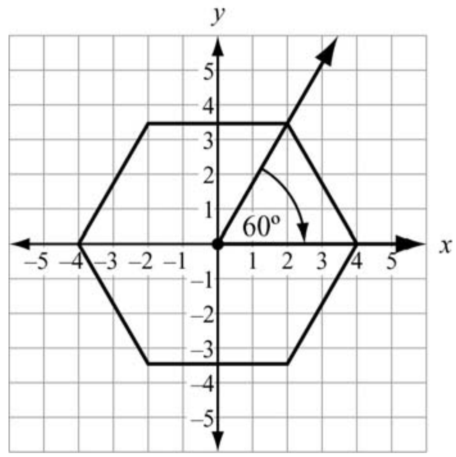
There are three transformations:

- Reflect across the line $y = 1$, or
- Reflect across the line $x = -2$, or
- Rotate 180° about the point (–2, 1).

4) Describe every transformation that maps this figure to itself: a regular hexagon (6 sides) centered about the origin, which has a vertex at (4, 0).

Solution:

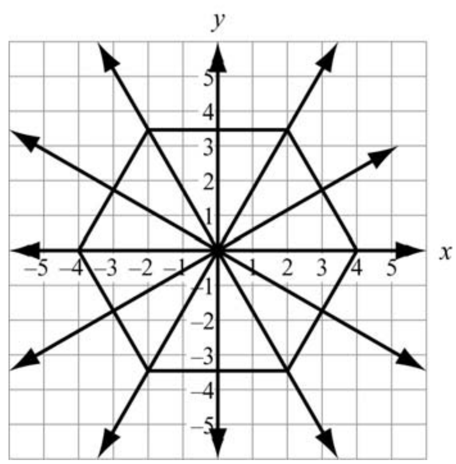
The angle formed by any two consecutive vertices and the center of the hexagon measures 60° because $\frac{360°}{6} = 60°$. So a rotation about the origin, clockwise or counterclockwise, of 60°, 120°, or any other multiple of 60° maps the hexagon to itself.



If a reflection across a line maps a figure to itself, then that line is called a ***line of symmetry***.

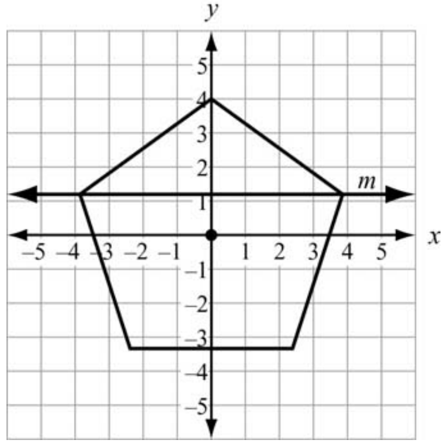
A regular hexagon has 6 lines of symmetry: 3 lines through opposite vertices and 3 lines through midpoints of opposite sides.

A reflection across any of the 6 lines of symmetry maps the hexagon to itself.



EOCT Practice Items

1) A regular pentagon is centered about the origin and has a vertex at (0, 4).

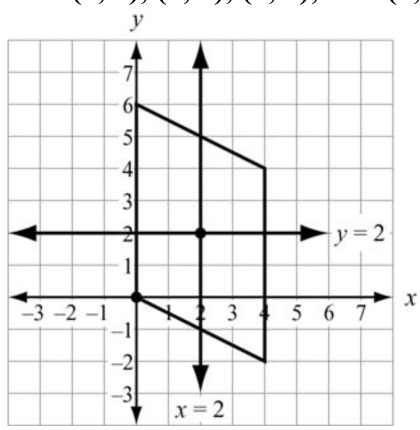


Which transformation maps the pentagon to itself?

A. a reflection across line *m*
B. a reflection across the *x*-axis
C. a clockwise rotation of 100° about the origin
D. a clockwise rotation of 144° about the origin

[Key: D]

2) A parallelogram has vertices at (0, 0), (0, 6), (4, 4), and (4, –2).

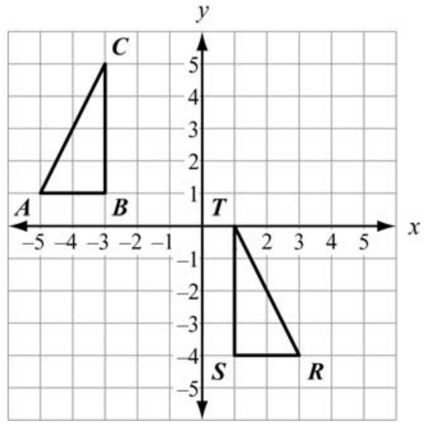


Which transformation maps the parallelogram to itself?

A. a reflection across the line $x = 2$
B. a reflection across the line $y = 2$
C. a rotation of 180° about the point (2, 2)
D. a rotation of 180° about the point (0, 0)

[Key: C]

3) Which sequence of transformations maps $\triangle ABC$ to $\triangle RST$?



A. Reflect $\triangle ABC$ across the line $x = -1$. Then translate the result 1 unit down.
B. Reflect $\triangle ABC$ across the line $x = -1$. Then translate the result 5 units down.
C. Translate $\triangle ABC$ 6 units to the right. Then rotate the result 90° clockwise about the point (1, 1).
D. Translate $\triangle ABC$ 6 units to the right. Then rotate the result 90° counterclockwise about the point (1, 1).

[Key: B]

Unit 6: Connecting Algebra and Geometry Through Coordinates

The focus of this unit is to have students analyze and prove geometric properties by applying algebraic concepts and skills on a coordinate plane. Students learn how to prove the fundamental theorems involving parallel and perpendicular lines and their slopes, applying both geometric and algebraic properties in these proofs. They also learn how to prove other theorems, applying to figures with specified numerical coordinates. (A theorem is any statement that is proved or can be proved. Theorems can be contrasted with postulates, which are statements that are accepted without proof.)

KEY STANDARDS

Use coordinates to prove simple geometric theorems algebraically.

MCC9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. For *example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point* (*0, 2*). (*Restrict contexts that use distance and slope*.)

MCC9-12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MCC9-12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

MCC9-12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.★

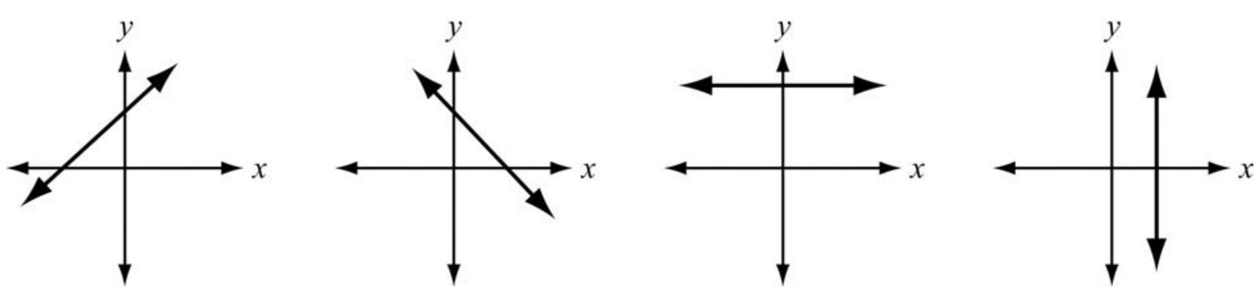
USE COORDINATES TO PROVE SIMPLE GEOMETRIC THEOREMS ALGEBRAICALLY

KEY IDEAS

1. The ***distance*** between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. The ***slope*** of the line through points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

3. Slopes can be positive, negative, 0, or undefined.



A line with a ***positive slope*** slants up to the right.

A line with a ***negative slope*** slants down to the right.

A line with a ***slope of 0*** is horizontal.

A line with an ***undefined slope*** is vertical.

4. Lines and their slopes are related by the following properties:
   a. Two nonvertical lines are ***parallel*** if and only if they have equal slopes.
   b. Two nonvertical lines are ***perpendicular*** if and only if the product of their slopes is –1.

5. Some useful ***properties of proportions*** state that all of the following are equivalent:

$$\frac{a}{b} = \frac{c}{d} \qquad ad = bc \qquad \frac{a}{c} = \frac{b}{d}$$

Example:

Use the multiplication property of equality to multiply each side of the proportion.

Because $\frac{5}{3}=\frac{10}{6}$, you can also write $5 \bullet 6 = 3 \bullet 10$ and $\frac{5}{10}=\frac{3}{6}$.

Multiply each side by 18, a common multiple of 3 and 6.

$$\frac{5}{3}(18)=(18)\frac{10}{6}$$

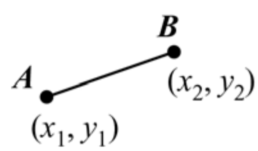
$$5 \bullet 6 = 3 \bullet 10$$

Multiply each side by $\frac{3}{10}$.

$$\frac{3}{10} \bullet \frac{5}{3} = \frac{3}{10} \bullet \frac{10}{6}$$

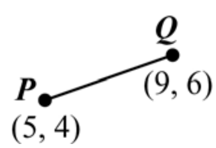
$$\frac{5}{10}=\frac{3}{6}$$

6. A ***directed line segment*** is a line segment from one point to another point in the coordinate plane. The ***components*** of directed line segment $\overline{AB}$ shown below are $(x_2 - x_1,\ y_2 - y_1)$. The components describe the direction and length of the directed line segment.



Example:

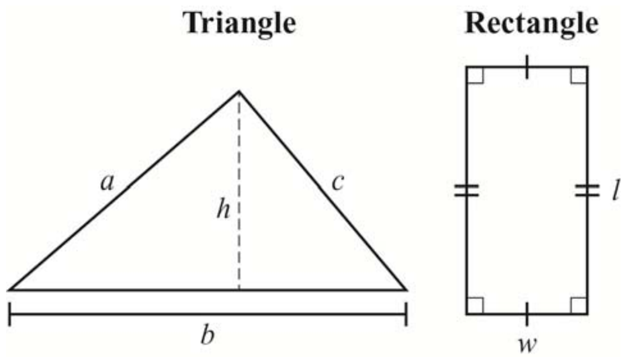
The components of $\overline{PQ}$ are $(9-5,\ 6-4)=(4,\ 2)$. They tell you that a "route" from *P* to *Q* is 4 units right and 2 units up. Note that the components are used in the slope: $\frac{6-4}{9-5}=\frac{2}{4}=\frac{1}{2}$.



Important Tip

- When using directed line segments, pay close attention to the beginning and end points of the line. For example, the directed line segments $\overline{PQ}$ and $\overline{QP}$ have the same length, but different directions.

7. The ***perimeter*** of a polygon is the sum of the lengths of the sides. The ***area*** of a polygon is the number of square units enclosed by the polygon.



For a triangle with side lengths a, b, c, with side b as the base and height h, the perimeter P and area A are:

$$P = a + b + c$$

$$A = \frac{1}{2}bh$$

For a rectangle with length l and width w, the perimeter P and area A are:

$$P = 2l + 2w$$

$$A = lw$$

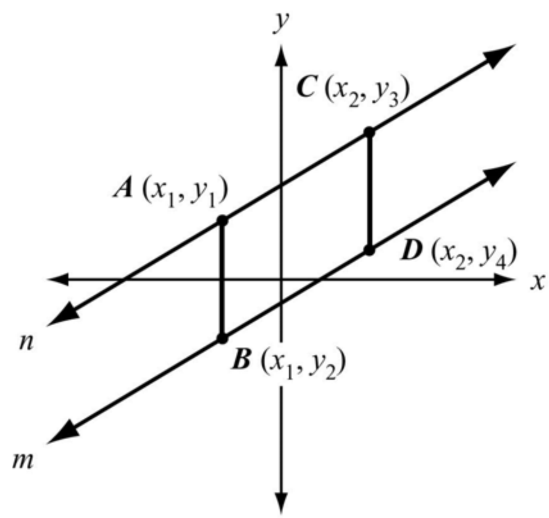
Important Tip

- In a triangle, any side can be used as the base. The corresponding height is the altitude drawn to the line containing that base. In a right triangle, the legs can be used as the base and height.

REVIEW EXAMPLES

1) Prove that if two nonvertical lines are parallel, then they have equal slopes.

Solution:



Assume that lines *n* and *m* are parallel. Points *A* and *C* lie on line *n*, and points *B* and *D* lie on line *m*.

Draw segments *AB* and *CD*. Both of these segments are vertical, so they are parallel. Lines *n* and *m* are parallel, so segments *AC* and *BD* are parallel. Because figure *BACD* has two pairs of parallel sides, it is a parallelogram. The opposite sides have the same length. Therefore, *AB* = *CD*.

$$AB = \sqrt{(x_1 - x_1)^2 + (y_2 - y_1)^2} \qquad CD = \sqrt{(x_2 - x_2)^2 + (y_4 - y_3)^2}$$

$$= \sqrt{(y_2 - y_1)^2} \qquad\qquad = \sqrt{(y_4 - y_3)^2}$$

$$= y_2 - y_1 \qquad\qquad = y_4 - y_3$$

Because *AB* = *CD*, $y_2 - y_1 = y_4 - y_3$. Use the subtraction property of equality to rewrite this as $y_3 - y_1 = y_4 - y_2$.

Now find the slopes of lines *n* and *m* using points *A*, *B*, *C*, and *D*.

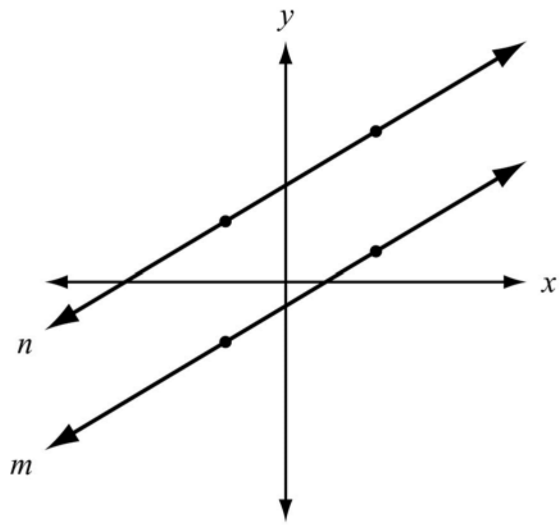
$$\text{slope of line } n = \frac{y_3 - y_1}{x_2 - x_1} \qquad \text{slope of line } m = \frac{y_4 - y_2}{x_2 - x_1}$$

Using substitution and the reflexive property, the slope of *n* is equal to the slope of *m*.

So, if two nonvertical lines are parallel, they have equal slopes.

2) Prove that if two nonvertical lines have equal slopes, then they are parallel.

Solution:



Use a proof by contradiction. Assume that the lines have equal slopes but are not parallel—that is, assume the lines intersect. If you can show this is not true, it is equivalent to proving the original statement.

Write the equations for both lines. The slopes are the same, so use m for the slope of each line. The two lines are different, so $b_1 \neq b_2$.

Equation for line n: $y = mx + b_1$
Equation for line m: $y = mx + b_2$

Solve a system of equations to find the point of intersection. Both equations are solved for y, so use substitution.

$$mx + b_1 = mx + b_2$$
$$b_1 = b_2$$

Because it was assumed that $b_1 \neq b_2$, this is a contradiction. So the original statement is true—if two nonvertical lines have equal slopes, then they are parallel.

3) The line p is represented by the equation $y = 4x + 1$. What is the equation of the line that is perpendicular to line p and passes through the point (8, 5)?

Solution:

Identify the slope of the line perpendicular to line p. The slopes of perpendicular lines are opposite reciprocals. Since the slope of line p is 4, $m = -\frac{1}{4}$.

The slope-intercept form of the equation of a line is $y = mx + b$. Substitute $-\frac{1}{4}$ for m. The line perpendicular to line p passes through (8, 5), so substitute 8 for x and 5 for y. Solve for b.

$$5 = -\frac{1}{4}(8) + b$$
$$5 = -2 + b$$
$$7 = b$$

The equation of the line perpendicular to line p and that passes through (8, 5) is $y = -\frac{1}{4}x + 7$.

4) For what value of n are the lines $7x + 3y = 8$ and $nx + 3y = 8$ perpendicular?

Solution:

The two lines will be perpendicular when the slopes are opposite reciprocals.

First, find the slope of the line $7x + 3y = 8$.

$$7x + 3y = 8$$
$$3y = -7x + 8$$
$$y = -\frac{7}{3}x + \frac{8}{3}$$

The slope is $-\frac{7}{3}$.

Next, find the slope of the line $nx + 3y = 8$, in terms of n.

$$nx + 3y = 8$$
$$3y = -nx + 8$$
$$y = -\frac{n}{3} + \frac{8}{3}$$

The slope is $-\frac{n}{3}$.

The opposite reciprocal of $-\frac{7}{3}$ is $\frac{3}{7}$. Find the value of n that makes the slope of the second line $\frac{3}{7}$.

$$-\frac{n}{3}=\frac{3}{7}$$
$$-n=\frac{9}{7}$$
$$n=-\frac{9}{7}$$

When $n=-\frac{9}{7}$, the two lines are perpendicular.

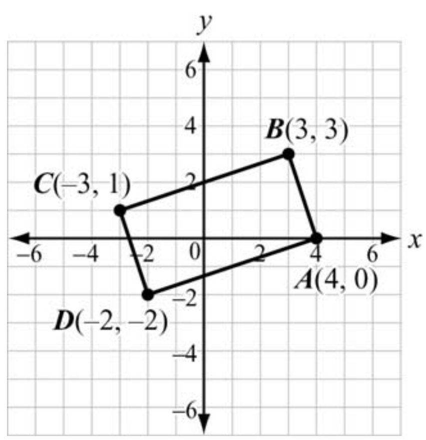
5) Quadrilateral *ABCD* has vertices *A*(4, 0), *B*(3, 3), *C*(–3, 1), and *D*(–2, –2). Prove that *ABCD* is a rectangle.

Solution:

The slopes of the sides are:

$\overline{AB}:\frac{3-0}{3-4}=\frac{3}{-1}=-3$ $\quad$ $\overline{BC}:\frac{1-3}{-3-3}=\frac{-2}{-6}=\frac{1}{3}$

$\overline{CD}:\frac{-2-1}{-2+3}=\frac{-3}{1}=-3$ $\quad$ $\overline{DA}:\frac{0+2}{4+2}=\frac{2}{6}=\frac{1}{3}$



$\overline{AB} \parallel \overline{CD}$ because they have equal slopes.

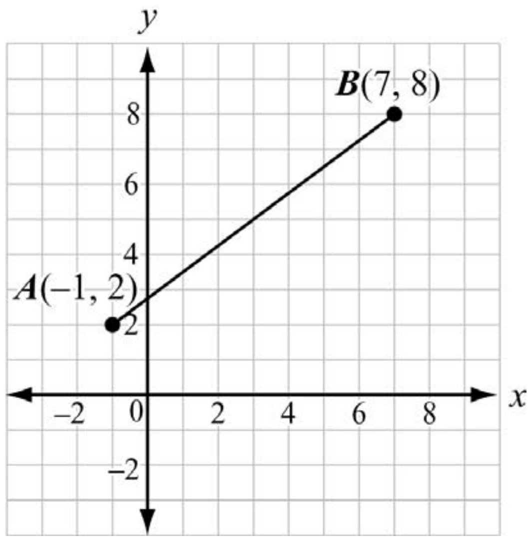
$\overline{BC} \parallel \overline{DA}$ because they have equal slopes.

So *ABCD* is a parallelogram because both pairs of opposite sides are parallel.

$\overline{AB} \perp \overline{BC}$ because the product of their slopes is –1: $-3 \bullet \frac{1}{3} = -1$.

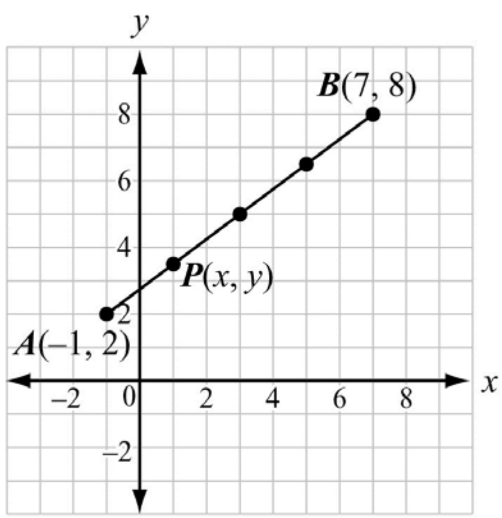
Therefore, *ABCD* is a rectangle because it is a parallelogram with a right angle.

6) Given the points *A*(–1, 2) and *B*(7, 8), find the coordinates of the point *P* on directed line segment $\overline{AB}$ that partitions $\overline{AB}$ in the ratio $\frac{1}{3}$.



Solution:

Point *P* partitions $\overline{AB}$ in the ratio $\frac{1}{3}$ if *P* is on $\overline{AB}$ and $\overline{AP} = \frac{1}{3}\overline{PB}$. This means that you need to split $\overline{AB}$ into 4 equal parts, since $\overline{AP} = \frac{1}{4}\overline{AB}$.



Let $P(x, y)$ be on $\overline{AB}$. Use components and solve two equations to find x and y, where (x_1, y_1) is the starting point, (x_2, y_2) is the ending point, and k is the fraction of the line segment (in this case, $\frac{1}{4}$).

$$(x, y) = \left(x_1 + k\left(x_2 - x_1\right),\ \ y_1 + k\left(y_2 - y_1\right)\right)$$

$$(x, y) = \left(-1 + \frac{1}{4}(7 - (-1)),\ \ 2 + \frac{1}{4}(8 - 2)\right)$$

$$x = -1 + \frac{1}{4}(7 - (-1)) \qquad y = 2 + \frac{1}{4}(8 - 2)$$

$$x = -1 + \frac{1}{4}(8) \qquad y = 2 + \frac{1}{4}(6)$$

$$x = -1 + 2 \qquad y = 2 + \frac{3}{2}$$

$$x = 1 \qquad y = \frac{7}{2}$$

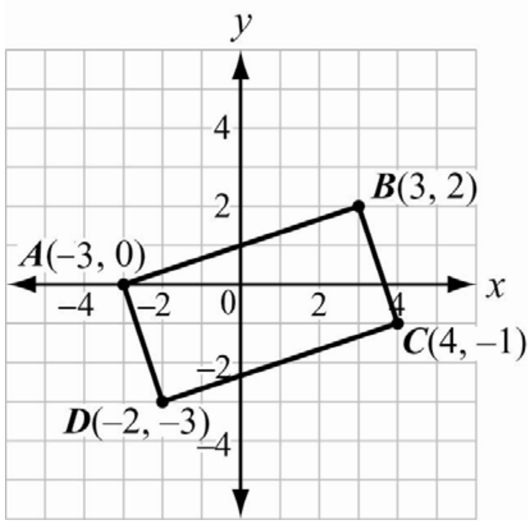
The coordinates of P are $\left(1, \frac{7}{2}\right)$.

Important Tip

- Be careful when using directed line segments. If point P partitions $\overline{AB}$ in the ratio $\frac{1}{3}$, then point P partitions $\overline{BA}$ in the ratio $\frac{3}{1}$.

7) Find the area of rectangle *ABCD* with vertices *A*(–3, 0), *B*(3, 2), *C*(4, –1), and *D*(–2, –3).

Solution:



Use the distance formula, $d=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}$, to find the length and width of the rectangle.

$AB=\sqrt{\left(3-(-3)\right)^2+\left(2-0\right)^2}=\sqrt{36+4}=\sqrt{40}$

$BC=\sqrt{\left(4-3\right)^2+\left(-1-2\right)^2}=\sqrt{1+9}=\sqrt{10}$

The length of the rectangle is usually considered to be the longer side. Therefore, the length of the rectangle is $\sqrt{40}$ and the width is $\sqrt{10}$.

Use the area formula.

$A=lw$

$A=(\sqrt{40})(\sqrt{10})$

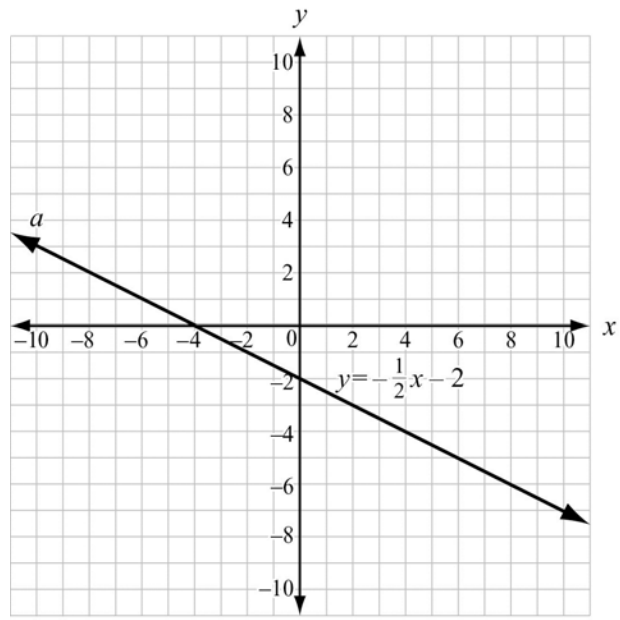
$A=\sqrt{400}$

$A=20$

The area of the rectangle is 20 square units.

EOCT Practice Items

1) An equation of line *a* is $y = -\frac{1}{2}x - 2$.



Which is an equation of the line that is perpendicular to line *a* and passes through the point (–4, 0)?

A. $y = -\frac{1}{2}x + 2$

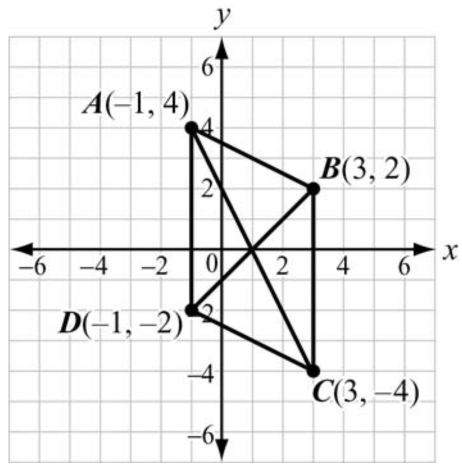
B. $y = -\frac{1}{2}x + 8$

C. $y = 2x - 2$

D. $y = 2x + 8$

[Key: D]

2) Parallelogram *ABCD* has vertices as shown.



Which equation would be used in proving that the diagonals of parallelogram *ABCD* bisect each other?

A. $\sqrt{(3-1)^2+(2-0)^2}=\sqrt{(1-3)^2+(0+4)^2}$

B. $\sqrt{(3+1)^2+(2+0)^2}=\sqrt{(1+3)^2+(0-4)^2}$

C. $\sqrt{(-1-1)^2+(4-0)^2}=\sqrt{(1-3)^2+(0+4)^2}$

D. $\sqrt{(-1+1)^2+(4+0)^2}=\sqrt{(1+3)^2+(0-4)^2}$

[Key: C]

3) Given the points *P*(2, –1) and *Q*(–9, –6), what are the coordinates of the point on directed line segment $\overline{PQ}$ that partitions $\overline{PQ}$ in the ratio $\frac{3}{2}$?

A. $\left(-\frac{23}{5},-4\right)$

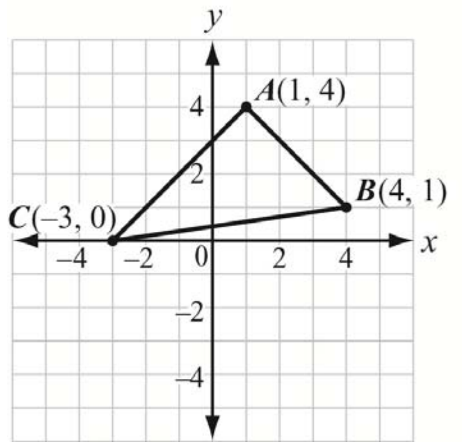
B. $\left(-\frac{12}{5},-3\right)$

C. $\left(-\frac{5}{3},-\frac{8}{3}\right)$

D. $\left(-\frac{5}{3},-\frac{8}{3}\right)$

[Key: A]

4) Triangle *ABC* has vertices as shown.



What is the area of the triangle?

A. $\sqrt{72}$ square units

B. 12 square units

C. $\sqrt{288}$ square units

D. 24 square units

[Key: B]

Appendix A
EOCT Sample Overall Study Plan Sheet

Here is a sample of what an OVERALL study plan might look like. You can use the Blank Overall Study Plan Sheet in Appendix B or create your own.

Materials/Resources I May Need When I Study:
(You can look back at page 6 for ideas.)

1. *This study guide*
2. *Pens/pencils*
3. *Highlighter*
4. *Notebook*
5. *Dictionary*
6. *Calculator*
7. *Mathematics textbook*

Possible Study Locations:

- First choice: *The library*
- Second choice: *My room*
- Third choice: *My mom's office*

Overall Study Goals:

1. *Read and work through the entire study guide.*
2. *Answer the sample questions and study the answers.*
3. *Do additional reading in a mathematics textbook.*

Number of Weeks I Will Study: *6 weeks*

Number of Days a Week I Will Study: *5 days a week*

Best Study Times for Me:

- Weekdays: *7:00 p.m. – 9:00 p.m.*
- Saturday: *9:00 a.m. – 11:00 a.m.*
- Sunday: *2:00 p.m. – 4:00 p.m.*

Appendix B
Blank Overall Study Plan Sheet

Materials/Resources I May Need When I Study:
(You can look back at page 6 for ideas.)

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

Possible Study Locations:

- First choice: ______________________________
- Second choice ______________________________
- Third choice ______________________________

Overall Study Goals:

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

Number of Weeks I Will Study: ______________________

Number of Days a Week I Will Study: ____________________

Best Study Times for Me:

- Weekdays: ______________________________
- Saturday: ______________________________
- Sunday: ______________________________

Appendix C
EOCT Sample Daily Study Plan Sheet

Here is a sample of what a DAILY study plan might look like. You can use the Blank Daily Study Plan Sheet in Appendix D or create your own.

Materials I May Need Today:

1. *Study guide*
2. *Pens/pencils*
3. *Notebook*

Today's Study Location: *The desk in my room*

Study Time Today: *From 7:00 p.m. to 8:00 p.m. with a short break at 7:30 p.m.*
(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. "Doing time" at your desk doesn't count as real studying.)

If I Start to Get Tired or Lose Focus Today, I Will: *Do some sit-ups*

Today's Study Goals and Accomplishments: (Be specific. Include things like number of pages, units, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small "chunks" or blocks of material at a time.)

Study Task	*Completed*	*Needs More Work*	*Needs More Information*
1. Review what I learned last time	X		
2. Study the first main topic in Unit 1	X		
3. Study the second main topic in Unit 1		X	

What I Learned Today:

1. *Reviewed basic functions*
2. *The importance of checking that the answer "makes sense" by estimating first*
3. *How to use math symbols*

Today's Reward for Meeting My Study Goals: *Eating some popcorn*

Appendix D
Blank Daily Study Plan Sheet

Materials I May Need Today:

1. ________________________________
2. ________________________________
3. ________________________________
4. ________________________________
5. ________________________________

Today's Study Location: ____________________

Study Time Today: ____________________
(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. "Doing time" at your desk doesn't count as real studying.)

If I Start To Get Tired or Lose Focus Today, I Will: ______________________________

Today's Study Goals and Accomplishments: (Be specific. Include things like number of pages, sections, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small "chunks" or blocks of material at a time.)

Study Task	*Completed*	*Needs More Work*	*Needs More Information*
1.			
2.			
3.			
4.			
5.			

What I Learned Today:

1. ____
2. ____
3. ____

Today's Reward for Meeting My Study Goals: ____________________________________