

Georgia Milestones

Assessment System



**Study/Resource Guide
for Students and Parents**

Coordinate Algebra



The Study/Resource Guides are intended to serve as a resource for parents and students. They contain practice questions and learning activities for the course. The standards identified in the Study/Resource Guides address a sampling of the state-mandated content standards.

For the purposes of day-to-day classroom instruction, teachers should consult the wide array of resources that can be found at www.georgiastandards.org.

Study/Resource Guide

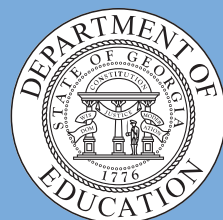


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THE GEORGIA MILESTONES ASSESSMENT SYSTEM



Dear Student,

The **Georgia Milestones Coordinate Algebra EOC Study/Resource Guide for Students and Parents** is intended as a resource for parents and students.

This guide contains information about the core content ideas and skills that are covered in the course. There are practice sample questions for every unit. The questions are fully explained and describe why each answer is either correct or incorrect. The explanations also help illustrate how each question connects to the Georgia state standards.

In addition, the guide includes activities that you can try to help you better understand the concepts taught in the course. The standards and additional instructional resources can be found on the Georgia Department of Education website, www.georgiastandards.org.

Get ready—open this guide—and get started!

GEORGIA MILESTONES END-OF-COURSE (EOC) ASSESSMENTS

The EOC assessments serve as the final exam in certain courses. The courses are:

English Language Arts

- American Literature and Composition

Mathematics

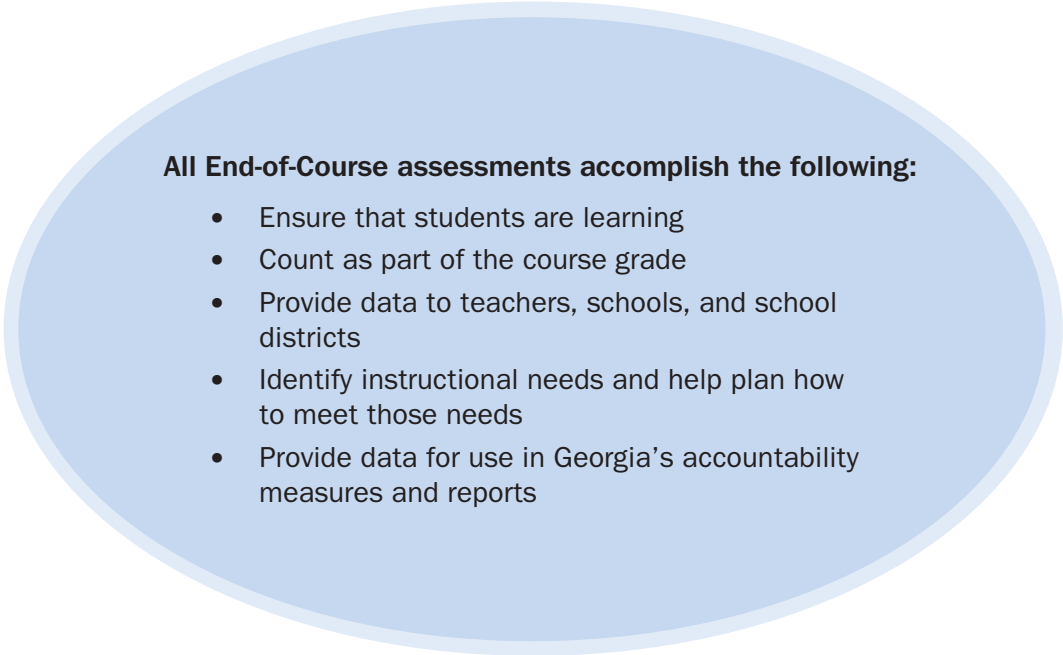
- Algebra I
- Coordinate Algebra

Science

- Biology

Social Studies

- United States History



All End-of-Course assessments accomplish the following:

- Ensure that students are learning
- Count as part of the course grade
- Provide data to teachers, schools, and school districts
- Identify instructional needs and help plan how to meet those needs
- Provide data for use in Georgia's accountability measures and reports

HOW TO USE THIS GUIDE

Let's get started!

First, preview the entire guide. Learn what is discussed and where to find helpful information. Even though the focus of this guide is Coordinate Algebra, you need to keep in mind your overall good reading habits.

- 💡 Start reading with a pencil or a highlighter in your hand and sticky notes nearby.
- 💡 Mark the important ideas, the things you might want to come back to, or the explanations you have questions about. On that last point, your teacher is your best resource.
- 💡 You will find some key ideas and important tips to help you prepare for the test.
- 💡 You can learn about the different types of items on the test.
- 💡 When you come to the sample items, don't just read them, *do* them. Think about strategies you can use for finding the right answer. Then read the analysis of the item to check your work. The reasoning behind the correct answer is explained for you. It will help you see any faulty reasoning in the ones you may have missed.
- 💡 Use the activities in this guide to get hands-on understanding of the concepts presented in each unit.
- 💡 With the Depth of Knowledge (DOK) information, you can gauge just how complex the item is. You will see that some items ask you to recall information and others ask you to infer or go beyond simple recall. The assessment will require all levels of thinking.
- 💡 Plan your studying and schedule your time.
- 💡 Proper preparation will help you do your best!




OVERVIEW OF THE COORDINATE ALGEBRA EOC ASSESSMENT

ITEM TYPES

The Coordinate Algebra EOC assessment consists of **selected-response** and **technology-enhanced** items.

A **selected-response** item, sometimes called a multiple-choice item, is a question, problem, or statement that is followed by four answer choices. These questions are worth one point.

A **technology-enhanced (TE)** item has a question, problem, or statement. These types of items are worth one or two points. Partial credit may be awarded on two-point items if you select some but not all of the correct answers or if you get one part of the question correct but not the other part.

- In multi-select items, you will be asked to select more than one right answer.
- In multi-part items, the items will have more than one part. You will need to provide an answer in each part.
- In drag-and-drop items, you will be asked to use a mouse, touchpad, or touchscreen to move responses to designated areas on the screen.
- In drop-down menu items, you will be asked to use a mouse, touchpad, or touchscreen to open a drop-down menu and select an option from the menu. A drop-down item may have multiple drop-down menus.
- In keypad-input items, you will be asked to use a physical keyboard or the pop-up keyboard on a touchscreen to type a number, expression, or equation into an answer box.
- In coordinate-graph items, you will be asked to use a mouse, touchpad, or touchscreen to draw lines and/or plot points on a coordinate grid on the screen.
- In line-plot items, you will be asked to use a mouse, touchpad, or touchscreen to place Xs above a number line to create a line plot.
- In bar-graph items, you will be asked to use a mouse, touchpad, or touchscreen to select the height of each bar to create a bar graph.
- In number-line items, you will be asked to use a mouse, touchpad, or touchscreen to plot a point and/or represent inequalities.
- Since some technology-enhanced items in this guide were designed to be used in an online, interactive-delivery format, some of the item-level directions will not appear to be applicable when working within the format presented in this document (for example, “Move the clocks into the graph” or “Create a scatter plot”).
- This icon  identifies special directions that will help you answer technology-enhanced items as shown in the format presented within this guide. These directions do not appear in the online version of the test but explain information about how the item works that would be easily identifiable if you were completing the item in an online environment.

To practice using technology-enhanced items in an online environment very similar to how they will appear on the online test, visit “Experience Online Testing Georgia.”

1. Go to the website “Welcome to Experience Online Testing Georgia” (<http://gaexperienceonline.com/>).
2. Select “Test Practice.”
3. Under “End-of-Course (EOC) All Assessments,” select “Online Tools Training.”
4. Select “EOC Test Practice.”
5. Select “Technology Enhanced Items.”
6. You will be taken to a login screen. Use the username and password provided on the screen to log in and practice navigating technology-enhanced items online.

Please note that Google Chrome is the only supported browser for this public version of the online testing environment.

DEPTH OF KNOWLEDGE DESCRIPTORS

Items found on the Georgia Milestones assessments, including the Coordinate Algebra EOC assessment, are developed with a particular emphasis on the kinds of thinking required to answer questions. In current educational terms, this is referred to as Depth of Knowledge (DOK). DOK is measured on a scale of 1 to 4 and refers to the level of cognitive demand (different kinds of thinking) required to complete a task, or in this case, an assessment item. The following table shows the expectations of the four DOK levels in detail.

The DOK table lists the skills addressed in each level as well as common question cues. These question cues not only demonstrate how well you understand each skill but also relate to the expectations that are part of the state standards.

Level 1—Recall of Information

Level 1 generally requires that you identify, list, or define. This level usually asks you to recall facts, terms, concepts, and trends and may ask you to identify specific information contained in documents, maps, charts, tables, graphs, or illustrations. Items that require you to “describe” and/or “explain” could be classified as Level 1 or Level 2. A Level 1 item requires that you just recall, recite, or reproduce information.

Skills Demonstrated	Question Cues
<ul style="list-style-type: none"> • Make observations • Recall information • Recognize formulas, properties, patterns, processes • Know vocabulary, definitions • Know basic concepts • Perform one-step processes • Translate from one representation to another • Identify relationships 	<ul style="list-style-type: none"> • Find • List • Define • Identify; label; name • Choose; select • Compute; estimate • Express • Read from data displays • Order

Level 2—Basic Reasoning

Level 2 includes the engagement (use) of some mental processing beyond recalling or reproducing a response. A Level 2 “describe” and/or “explain” item would require that you go beyond a description or explanation of recalled information to describe and/or explain a result or “how” or “why.”

Skills Demonstrated	Question Cues
<ul style="list-style-type: none"> • Apply learned information to abstract and real-life situations • Use methods, concepts, and theories in abstract and real-life situations • Perform multi-step processes • Solve problems using required skills or knowledge (requires more than habitual response) • Make a decision about how to proceed • Identify and organize components of a whole • Extend patterns • Identify/describe cause and effect • Recognize unstated assumptions; make inferences • Interpret facts • Compare or contrast simple concepts/ideas 	<ul style="list-style-type: none"> • Apply • Calculate; solve • Complete • Describe • Explain how; demonstrate • Construct data displays • Construct; draw • Analyze • Extend • Connect • Classify • Arrange • Compare; contrast

Level 3—Complex Reasoning

Level 3 requires reasoning, using evidence, and thinking on a higher and more abstract level than Level 1 and Level 2. You will go beyond explaining or describing “how and why” to justifying the “how and why” through application and evidence. Level 3 items often involve making connections across time and place to explain a concept or a “big idea.”

Skills Demonstrated	Question Cues
<ul style="list-style-type: none"> Solve an open-ended problem with more than one correct answer Create a pattern Relate knowledge from several sources Draw conclusions Make predictions Translate knowledge into new contexts Assess value of methods, concepts, theories, processes, and formulas Make choices based on a reasoned argument Verify the value of evidence, information, numbers, and data 	<ul style="list-style-type: none"> Plan; prepare Predict Create; design Generalize Justify; explain why; support; convince Assess Rank; grade Test; judge Recommend Select Conclude

Level 4—Extended Reasoning

Level 4 requires the complex reasoning of Level 3 with the addition of planning, investigating, applying significant conceptual understanding, and/or developing that will most likely require an extended period of time. You may be required to connect and relate ideas and concepts *within* the content area or *among* content areas in order to be at this highest level. The Level 4 items would be a show of evidence, through a task, a product, or an extended response, that the cognitive demands have been met.

Skills Demonstrated	Question Cues
<ul style="list-style-type: none"> Analyze and synthesize information from multiple sources Apply mathematical models to illuminate a problem or situation Design a mathematical model to inform and solve a practical or abstract situation Combine and synthesize ideas into new concepts 	<ul style="list-style-type: none"> Design Connect Synthesize Apply concepts Critique Analyze Create Prove

DEPTH OF KNOWLEDGE EXAMPLE ITEMS

Example items that represent the applicable DOK levels across various Coordinate Algebra content domains are provided on the following pages.

All example and sample items contained in this guide are the property of the Georgia Department of Education.

Example Item 1

Selected-Response

DOK Level 1: This is a DOK Level 1 item because it asks students to recall information and determine which relationship does not have the properties that fit the definition of a function.

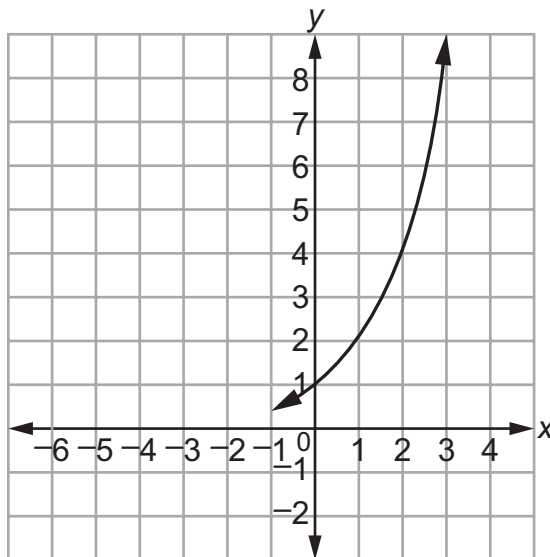
Coordinate Algebra Content Domain: Functions

Standard: MGSE9-12.F.IF.1. Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

Which of these is NOT a function?

A. $(5, 3), (6, 4), (7, 3), (8, 4)$

B.



C. $y = 3x$

D. $(6, 8), (3, 7), (4, 2), (3, 1)$

Correct Answer: D

Explanation of Correct Answer: The correct answer is choice (D). A function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range, but for the set of ordered pairs, there are two y -values for x when $x = 3$. Therefore, it does not meet the definition of a function. Choices (A), (B), and (C) are functions because for every x -value, there is only one y -value.

Example Item 2

Selected-Response

DOK Level 2: This is a DOK Level 2 item because it requires basic reasoning and asks students to apply their knowledge of functions that are undefined and extend that concept to determine the domain of this function.

Coordinate Algebra Content Domain: Functions

Standard: MGSE9-12.F.IF.1. Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of the domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

The number of school buses needed to transport students on a field trip is given by the function

$f(x) = \frac{x}{40}$ where x represents the number of students going on the trip. What is the domain of this function?

- A. x is the set of all real numbers.
- B. x is the set of all integers.
- C. x is the set of all nonnegative integers.
- D. x is the set of all nonnegative real numbers.

Correct Answer: C

Explanation of Correct Answer: The correct answer is choice (C), x is the set of all nonnegative integers. Choices (A), (B), and (D) would include either fractional numbers, negative numbers, or both. The number of students must be a positive and whole number.

Example Item 3

Keypad-Input Multi-Part Technology-Enhanced

DOK Level 3: This is a DOK level 3 item because it requires students to use multiple steps and multiple concepts to solve a contextual problem.

Coordinate Algebra Content Domain: Functions

Standard: MGSE9-12.F.IF.7a. Graph linear functions and show intercepts, maxima, and minima (as determined by the function or by context).


Part A

Michelle will sell her homemade necklaces for \$8 each at a craft fair. It costs her \$150 for the booth rental at the craft fair. She made 200 necklaces to sell.

Part A What is the maximum amount, in dollars, of profit Michelle can earn from the sale of her necklaces?



The keypad interface includes a top row with a yellow eraser icon, a left arrow, a right arrow, a double left arrow, a double right arrow, and a delete (X) icon. To the right of these icons is a question mark icon. Below this row is a large empty rectangular box for the answer. At the bottom left is a numeric keypad with digits 1-9, 0, a decimal point, and a negative sign. To the right of the numeric keypad is a fraction template icon.

 Use a mouse, touchpad, or touchscreen to enter a response.

Go on to the next page to finish example item 3.

Example Item 3. *Continued.*


Part B

Michelle will sell her homemade necklaces for \$8 each at a craft fair. It costs her \$150 for the booth rental at the craft fair. She made 200 necklaces to sell.

Part B Michelle wants to graph the equation for her profit from selling her necklaces. The y -intercept of this graph will be her profit from selling none of her necklaces.

What is the y -intercept of the graph of the equation that represents Michelle's profit from selling her necklaces?

1	2	3	n/a
4	5	6	
7	8	9	
0	.	(-)	

 Use a mouse, touchpad, or touchscreen to enter a response.

Go on to the next page to finish example item 3.

Example Item 3

Scoring Rubric

Points	Description
2	The student correctly answers both Part A and Part B.
1	The student correctly answers either Part A OR Part B.
0	The student does not correctly answer either part.

Exemplar Response

Part A

The correct response is shown below.

A digital math input interface. At the top is a toolbar with icons for erasing, undo, redo, left arrow, right arrow, and a delete key (X). To the right of the toolbar is a question mark icon. Below the toolbar, the number "1450" is entered in a large text box. Below this is a numeric keypad with digits 1-9, 0, a decimal point, and a negative sign. To the right of the keypad is a fraction template icon.

This is the correct response because $8 \times 200 = \$1,600$, which is the maximum she can earn from selling the necklaces. Then, \$150, the cost for renting the booth, is subtracted from \$1,600 to find the profit.

Part B

The correct response is shown below.

A digital math input interface. At the top is a toolbar with icons for erasing, undo, redo, left arrow, right arrow, and a delete key (X). To the right of the toolbar is a question mark icon. Below the toolbar, the number "-150" is entered in a large text box. Below this is a numeric keypad with digits 1-9, 0, a decimal point, and a negative sign. To the right of the keypad is a fraction template icon.

This is the correct response because the y-intercept represents the sale of 0 bracelets, which results in \$0 from sales, but she still has to pay \$150 for the booth.

DESCRIPTION OF TEST FORMAT AND ORGANIZATION

The Georgia Milestones Coordinate Algebra EOC assessment consists of a total of 55 items. You will be asked to respond to selected-response (multiple-choice) and technology-enhanced items.

The test will be given in two sections.

- You may have up to 65 minutes per section to complete Sections 1 and 2.
- The total estimated testing time for the Coordinate Algebra EOC assessment ranges from approximately 60 to 130 minutes. Total testing time describes the amount of time you have to complete the assessment. It does not take into account the time required for the test examiner to complete pre-administration and post-administration activities (such as reading the standardized directions to students).
- Sections 1 and 2 may be administered on the same day or across two consecutive days, based on the district's testing protocols for the EOC measures (in keeping with state guidance).
- During the Coordinate Algebra EOC assessment, a formula sheet will be available for you to use. Another feature of the Coordinate Algebra assessment is that you may use a graphing calculator in calculator-approved sections.

Effect on Course Grade

It is important that you take this course and the EOC assessment very seriously.

- For students in Grade 10 or above beginning with the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 85% and the EOC score 15%.
- For students in Grade 9 beginning with the 2011–2012 school year and later, the final grade in each course is calculated by weighing the course grade 80% and the EOC score 20%.
- A student must have a final grade of at least 70% to pass the course and to earn credit toward graduation.




PREPARING FOR THE COORDINATE ALGEBRA EOC ASSESSMENT

STUDY SKILLS

As you prepare for this test, ask yourself the following questions:





- * How would you describe yourself as a student?
- * What are your study skills strengths and/or weaknesses?
- * How do you typically prepare for a classroom test?
- * What study methods do you find particularly helpful?
- * What is an ideal study situation or environment for you?
- * How would you describe your actual study environment?
- * How can you change the way you study to make your study time more productive?

ORGANIZATION—OR TAKING CONTROL OF YOUR WORLD

-  Establish a study area that has minimal distractions.
-  Gather your materials in advance.
-  Develop and implement your study plan.

ACTIVE PARTICIPATION

The most important element in your preparation is *you*. You and your actions are the key ingredient. Your active studying helps you stay alert and be more productive. In short, you need to interact with the course content. Here's how you do it.

-  Carefully read the information and then DO something with it. Mark the important material with a highlighter, circle it with a pen, write notes on it, or summarize the information in your own words.
-  Ask questions. As you study, questions often come into your mind. Write them down and actively seek the answers.
-  Create sample test questions and answer them.
-  Find a friend who is also planning to take the test and quiz each other.

TEST-TAKING STRATEGIES

Part of preparing for a test is having a set of strategies you can draw from. Include these strategies in your plan:

- * Read and understand the directions completely. If you are not sure, ask a teacher.
- * Read each question and all of the answer choices carefully.
- * If you use scratch paper, make sure you copy your work to your test accurately.
- * Underline the important parts of each task. Make sure that your answer goes on the answer sheet.
- * Be aware of time. If a question is taking too much time, come back to it later.
- * Answer all questions. Check your answers for accuracy.
- * Stay calm and do the best you can.

PREPARING FOR THE COORDINATE ALGEBRA EOC ASSESSMENT





Read this guide to help prepare for the Coordinate Algebra EOC assessment.

The section of the guide titled “Content of the Coordinate Algebra EOC Assessment” provides a snapshot of the Coordinate Algebra course. In addition to reading this guide, do the following to prepare to take the assessment:

- Read your resources and other materials.
- Think about what you learned, ask yourself questions, and answer them.
- Read and become familiar with the way questions are asked on the assessment.
- Answer some practice Coordinate Algebra questions.
- There are additional items to practice your skills available online. Ask your teacher about online practice sites that are available for your use.

CONTENT OF THE COORDINATE ALGEBRA EOC ASSESSMENT





Up to this point in the guide, you have been learning how to prepare for taking the EOC assessment. Now you will learn about the topics and standards that are assessed in the Coordinate Algebra EOC assessment and will see some sample items.

-  The first part of this section focuses on what will be tested. It also includes sample items that will let you apply what you have learned in your classes and from this guide.
-  The second part of this section contains additional items to practice your skills.
-  The next part contains a table that shows the standard assessed for each item, the DOK level, the correct answer (key), and a rationale/explanation of the right and wrong answers for the additional practice items.
-  You can use the sample items to familiarize yourself with the item formats found on the assessment.

All example and sample items contained in this guide are the property of the Georgia Department of Education.

The Coordinate Algebra EOC assessment will assess the Coordinate Algebra standards documented at www.georgiastandards.org.

The content of the assessment is organized into four groupings, or domains, of standards for the purpose of providing feedback on student performance.

-  A content domain is a reporting category that *broadly* describes and defines the content of the course, as measured by the EOC assessment.
-  On the actual test the standards for Coordinate Algebra are grouped into four domains that follow your classwork: Algebra (*includes Number and Quantity*), Functions, Algebra Connections to Geometry, and Algebra Connections to Statistics and Probability.
-  Each domain was created by organizing standards that share similar content characteristics.
-  The content standards describe the level of understanding each student is expected to achieve. They include the knowledge, concepts, and skills assessed on the EOC assessment, and they are used to plan instruction throughout the course.

SNAPSHOT OF THE COURSE

This section of the guide is organized into six units that review the material taught within the four domains of the Coordinate Algebra course. The material is presented by concept rather than by category or standard. In each unit you will find sample items similar to what you will see on the EOC assessment. The next section of the guide contains additional items to practice your skills followed by a table that shows for each item the standard assessed, the DOK level, the correct answer (key), and a rationale/explanation about the key and distractors.

All example and sample items contained in this guide are the property of the Georgia Department of Education.

The more you understand about the standard in each unit, the greater your chances of getting a good score on the EOC.

UNIT 1: RELATIONSHIPS BETWEEN QUANTITIES

In this unit, you will study quantitative relationships. You will learn how important units are for interpreting problems and setting up equations. The focus will be on both one- and two-variable linear and exponential equations. There will also be examples of modeling with inequalities.

1.1 Quantities and Units

MGSE9-12.N.Q.1 Use units of measure (linear, area, capacity, rates, and time) as a way to understand problems:

- Identify, use, and record appropriate units of measure within context, within data displays, and on graphs;
- Convert units and rates using dimensional analysis (English-to-English and Metric-to-Metric without conversion factor provided and between English and Metric with conversion factor);
- Use units within multistep problems and formulas; interpret units of input and resulting units of output.

MGSE9-12.N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. Given a situation, context, or problem, students will determine, identify, and use appropriate quantities for representing the situation.

MGSE9-12.N.Q.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. *For example, money situations are generally reported to the nearest cent (hundredth). Also, an answer's precision is limited to the precision of the data given.*

KEY IDEAS

A **quantity** is an exact amount or measurement. One type of quantity is a simple count, such as 5 eggs or 12 months. A second type of quantity is a measurement, which is an amount of a specific unit. Examples are 6 feet and 3 pounds.

A quantity can be exact or approximate. When an approximate quantity is used, it is important that we consider its level of accuracy. When working with measurements, we need to determine what level of accuracy is necessary and practical. For example, a dosage of medicine would need to be very precise. An example of a measurement that does not need to be very precise is the distance from your house to a local mall. The use of an appropriate unit for measurements is also important. For example, if you want to calculate the diameter of the Sun, you would want to choose a very large unit as your measure of length, such as miles or kilometers. Conversion of units can require approximations.

Example: Convert 309 yards to feet.

We know 1 yard is 3 feet, which we can write as a fraction $\frac{3 \text{ feet}}{1 \text{ yard}}$.

$$309 \text{ yards} \times \frac{3 \text{ feet}}{1 \text{ yard}} = 927 \text{ feet}$$

Since the multiplication contains yards in the numerator and denominator, yards will cancel. We can approximate that 309 yards is close to 900 feet.

The context of a problem tells us what types of units are involved. **Dimensional analysis** is a way to determine relationships among quantities using their dimensions, units, or unit equivalencies. Dimensional analysis suggests which quantities should be used for computation in order to obtain the desired result.

Example: The cost, in dollars, of a single-story home can be approximated using the formula $C = k/lw$, where l is the approximate length of the home and w is the approximate width of the home. Find the units for the coefficient k .

The coefficient k is a rate of cost, in dollars, for homes. To find the units for k , solve the equation $C = k/lw$, and then look at the units.

$$C \text{ dollars} = k \times l \text{ feet} \times w \text{ feet}$$

$$C = k/lw$$

$$\frac{C}{lw} = k$$

$$\frac{C \text{ dollars}}{lw \text{ feet} \cdot \text{feet}} = k$$

The value of k is $\frac{C}{lw}$, and the unit is dollars per feet squared or dollars per square foot.

You can check this using dimensional analysis:

$$C = k/lw$$

$$C = \frac{k \text{ dollars}}{\text{feet} \cdot \text{feet}} \cdot l \text{ feet} \cdot w \text{ feet}$$

$$C = k/lw \text{ dollars}$$

The process of dimensional analysis is also used to convert from one unit to another. Knowing the relationship between units is essential for unit conversion.

Example: Convert 45 miles per hour to feet per minute.

To convert the given units, we use a form of dimensional analysis. We will multiply 45 mph by a series of ratios where the numerator and denominator are in different units but equivalent to each other. The ratios are carefully chosen to introduce the desired units.

$$\frac{45 \text{ miles}}{1 \text{ hr}} \times \frac{1 \text{ hr}}{60 \text{ minutes}} \times \frac{5,280 \text{ feet}}{1 \text{ mile}} = \frac{45 \times 5,280 \text{ feet}}{60 \times 1 \text{ minute}} = 3,960 \text{ feet per minute}$$

When data are displayed in a graph, the units and scale features are keys to interpreting the data. Breaks or an abbreviated scale in a graph should be noted as they can cause a misinterpretation of the data.

The measurements we use are often approximations. It is routinely necessary to determine reasonable approximations.

Example: When Justin goes to work, he drives at an average speed of 60 miles per hour. It takes about 1 hour and 30 minutes for Justin to get to work. His car travels about 25 miles per gallon of gas. If gas costs \$3.65 per gallon, how much money does Justin spend on gas to get to work?

First, calculate the distance Justin travels.




$$60 \text{ miles per hour} \cdot 1.5 \text{ hour} = 90 \text{ miles}$$

Justin can travel 25 miles on 1 gallon of gas. Because 90 miles is close to 100 miles, he needs about $100 \div 25 = 4$ gallons of gas.

To find the cost of gas to get to work, multiply cost per gallon by the number of gallons.

$$4 \times \$3.65 = \$14.60$$

Important Tips

-  When referring to a quantity, include the unit or the items being counted whenever possible.
-  It is important to use appropriate units for measurements and to understand the relative sizes of units for the same measurement. You will need to know how to convert between units and how to round or limit the number of digits you use.
-  Use units to help determine whether your answer is reasonable. For example, if a question asks for a weight and you find an answer in feet, check your answer.

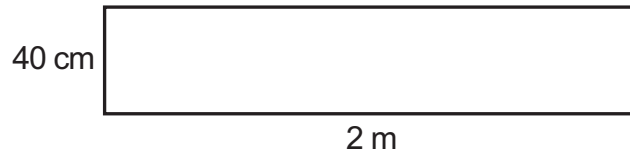
REVIEW EXAMPLES

- ◆ The formula for density d is $d = \frac{m}{v}$, where m is mass and v is volume. If mass is measured in kilograms and volume is measured in cubic meters, what is the unit for density?

Solution:

The unit for density is $\frac{\text{kilograms}}{\text{meters}^3}$, or $\frac{\text{kg}}{\text{m}^3}$.

- ◆ A rectangle has a length of 2 meters and a width of 40 centimeters. What is the perimeter of the rectangle?



Solution:

The perimeter can be found by adding all side lengths. The perimeter of a rectangle can also be found by using the formula $P = 2l + 2w$, where P is perimeter, l is length, and w is width.

To find the perimeter, both measurements need to have the same units. Convert 2 meters to centimeters or convert 40 centimeters to meters. Both methods are shown.

Cancel the like units and multiply the remaining factors. The product is the converted measurement.

Method 1

$$2 \text{ m} \times \frac{100 \text{ cm}}{1 \text{ m}} = 200 \text{ cm}$$

Method 2

$$40 \text{ cm} \times \frac{1 \text{ m}}{100 \text{ cm}} = 0.4 \text{ m}$$

Use the converted measurement in the formula to find the perimeter.

Method 1

$$\begin{aligned} P &= 2l + 2w \\ P &= 2(200) + 2(40) \\ P &= 400 + 80 \\ P &= 480 \text{ cm} \end{aligned}$$

Method 2

$$\begin{aligned} P &= 2l + 2w \\ P &= 2(2) + 2(0.4) \\ P &= 4 + 0.8 \\ P &= 4.8 \text{ m} \end{aligned}$$

SAMPLE ITEMS

1. A rectangle has a length of 12 meters and a width of 400 centimeters. What is the perimeter, in cm, of the rectangle?
 - A. 824
 - B. 1,600
 - C. 2,000
 - D. 3,200

2. Jill swam 200 meters in 2 minutes 42 seconds. If each lap is 50 meters long, which time is her estimated time, in seconds, per lap?
 - A. 32
 - B. 40
 - C. 48
 - D. 60

Answers to Unit 1.1 Sample Items

1. D 2. B

1.2 Structure of Expressions

MGSE9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.

MGSE9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients, in context.

MGSE9-12.A.SSE.1b Given situations which utilize formulas or expressions with multiple terms and/or factors, interpret the meaning (in context) of individual terms or factors.

KEY IDEAS

Arithmetic expressions contain numbers and operation signs. Examples: $2 + 4$, $4 - (10 + 3)$ and $\frac{\sqrt{9} + 2}{5}$

Algebraic expressions contain one or more variables. Examples: $2x + 4$, $4x - (10 + 3y)$, and $\frac{9 + 2t}{5}$

The parts of expressions that are separated by addition or subtraction signs are called **terms**. Terms are usually composed of numerical factors and variable factors. **Factors** are numbers multiplied together to get another number. A numerical factor is called a **coefficient**.

Consider the algebraic expression $4x + 7y - 3$. It has three terms: $4x$, $7y$, and -3 . For $4x$, the coefficient is 4 and the variable factor is x . For $7y$, the coefficient is 7 and the variable is y . The third term, -3 , has no variables and is called a **constant**.


To interpret a formula, it is important to know what each variable represents and to understand the relationships between the variables.

For example, look at the compound interest formula $A = P(1 + \frac{r}{n})^{nt}$, where A is the balance (the amount in the account), P is the principal (starting amount deposited), r is the interest rate, n is the number of times the interest is compounded per year, and t is the number of years.

This formula is used to calculate the balance of an account when the annual interest is compounded. The balance is the product of P and $(1 + \frac{r}{n})^{nt}$. Because the value $\frac{r}{n}$ is positive, the formula is a growth formula.

Another example is the formula used to estimate the number of calories burned while jogging, $C = 0.075mt$, where m represents a person's body weight in pounds and t is the number of minutes spent jogging. This formula tells us that the number of calories burned depends on a person's body weight and how much time is spent jogging. The coefficient, 0.075, is the factor used for jogging. Since sprinting burns more calories in less time, the coefficient for sprinting would be larger than 0.075.

Important Tip

 To consider how a coefficient affects a term, try different coefficient values for the same term and explore the effects.

REVIEW EXAMPLES

- ◆ An amount of \$1,000 is deposited into a bank account that pays 4% annual interest. This formula gives the amount of money in the account after t years.

$$A = 1,000(1 + 0.04)^t$$

How does $(1 + 0.04)$ in the equation affect the amount in the bank account?

Solution:

$(1 + 0.04)$ or 1.04 represents the growth factor of the amount of money in the bank account.

- ◆ The number of calories burned during exercise depends on the activity. The formulas for two activities are given.

$$C_1 = 0.012mt \text{ and } C_2 = 0.032mt$$

- If one activity is walking and the other is running, identify the formula that represents each activity. Explain your answer.
- What value would you expect the coefficient to have if the activity were reading? Include units and explain your answer.

Solution:

- The coefficient of the variable term mt tells us how strenuous the activity is. Since running is more strenuous than walking, its formula would have a higher coefficient. Therefore, the formula for running is likely $C_2 = 0.032mt$.
- Since reading is less strenuous than walking, the number of calories burned is probably fewer and the coefficient is probably smaller. Expect a coefficient smaller than 0.012 calorie per minute for reading.

SAMPLE ITEMS

1. The distance a car travels can be found using the formula $d = rt$, where d is the distance in miles, r is the rate of speed in miles per hour, and t is time in hours. Bill drives his car at 70 miles per hour for $\frac{1}{2}$ hour.

Which term gives the distance in miles Bill travels?

- A. 1
 - B. 2
 - C. $(2)(70)$
 - D. $(70)(\frac{1}{2})$
2. A certain population of bacteria has an average growth rate of 0.02 bacteria per hour. The formula for the growth of the bacteria's population is $A = P_0(2.71828)^{0.02t}$, where P_0 is the original population, and t is the time in hours.

Which factor represents the rate at which the bacteria grows?

- A. P_0
- B. $0.02t$
- C. 2.71828
- D. 1,478

Answers to Unit 1.2 Sample Items

1. D 2. C

1.3 Creating Equations and Inequalities

MGSE9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and exponential functions (integer inputs only).

MGSE9-12.A.CED.2 Create linear, quadratic, and exponential equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (The phrase “in two or more variables” refers to formulas like the compound interest formula, in which $A = P(1 + \frac{r}{n})^{nt}$ has multiple variables.)

MGSE9-12.A.CED.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret data points as possible (i.e. a solution) or not possible (i.e. a non-solution) under the established constraints.

MGSE9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest using the same reasoning as in solving equations. *Examples: Rearrange Ohm’s law $V = IR$ to highlight resistance R .*

KEY IDEAS

Problems in quantitative relationships generally call for us to determine how the quantities are related to each other, use a variable or variables to represent unknowns, and write and solve an equation. Some problems can be modeled using an equation with one unknown.

Example: The sum of the angle measures in a triangle is 180° . Two angles of a triangle measure 30° and 70° . What is the measure of the third angle?

The sum of the angle measures in a triangle is 180° . Let x° represent the measure of the third angle.

$$30^\circ + 70^\circ + x^\circ = 180^\circ$$

Next, we solve for x .

$$30 + 70 + x = 180 \quad \text{Write the original equation.}$$

$$100 + x = 180 \quad \text{Combine like terms.}$$

$$x = 80 \quad \text{Subtract 100 from both sides.}$$

The third angle measures 80° .

There are also problems that can be modeled with inequalities. These can be cases where you want to find the minimum or maximum amount of something.

Example: A social media website currently has 1,000 members. The number of members on the website triples every month. After how many months will the website first have more than 1,000,000 members?

To triple means to multiply by 3. To find the number of months, x , it will take for the number of members to triple, find the number of factors of 3 that should be multiplied by 1,000 so that the product is more than 1,000,000. This can be written as an inequality:

$$1,000 \cdot 3^x > 1,000,000$$

To solve, first divide both sides by 1,000 to get $3^x > 1,000$.

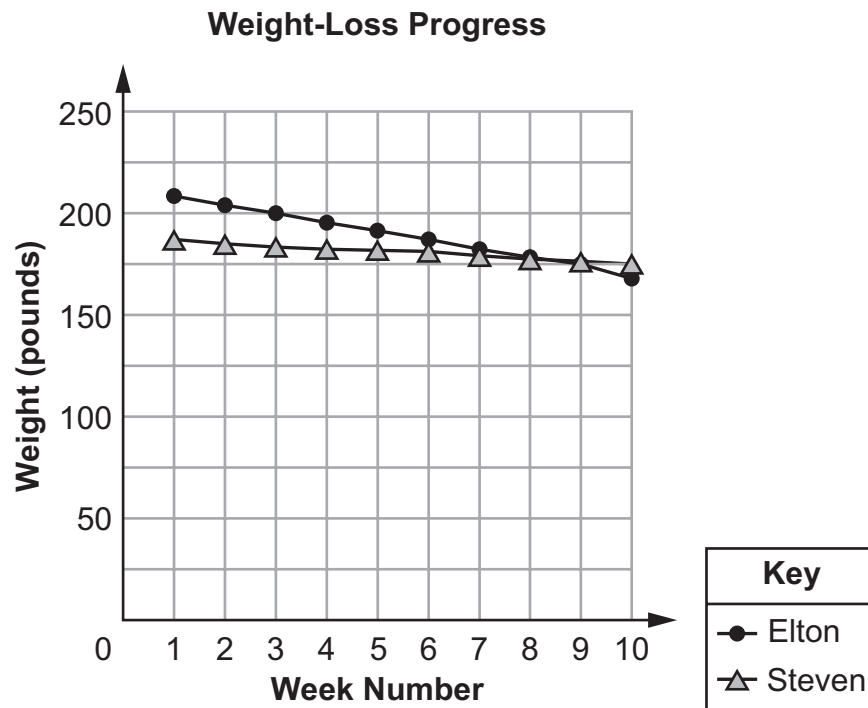
Next, determine the number of times to multiply 3 by itself so that the product is greater than 1,000. One way to do this is to use a chart.

3^1	3^2	3^3	3^4	3^5	3^6	3^7
3	9	27	81	243	729	2,187

Because $3^7 > 1,000$, it will take 7 months for there to be more than 1,000,000 members.

Some situations involve a pair of related variables. There are situations when two related variables can be modeled with equations and inequalities. Related variables can be interpreted using data points to show possible solutions.

Example: Elton loses 5 pounds each week. He started at 218 pounds in Week 1. Steven was 186 pounds in Week 1, and he loses 1 pound each week. The graph shows Elton's and Steven's weights by week.



- What equations can be used to represent Elton's and Steven's weight loss?
 - After how many weeks do both Elton and Steven weigh the same number of pounds?
- Let p represent weight in pounds, and let t represent the number of weeks. Elton's weight can be represented by the equation $p = 218 - 5t$. Steven's weight can be represented by the equation $p = 186 - t$.
 - The graph seems to show that the lines intersect at week 8. To make sure this is correct, find the value of each equation for $t = 8$.
 Elton: $218 - 5 \times 8 = 178$ pounds
 Steven: $186 - 1 \times 8 = 178$ pounds
 After 8 weeks, Elton and Steven both weigh 178 pounds.

There are situations with two related variables that can be modeled with inequalities. These inequalities are called **constraints**.

Example: Mark has \$14 to buy lunch for himself and his sister. He wants to buy at least 1 sandwich and 1 drink. Sandwiches cost \$5, and drinks cost \$2. Write three inequalities to represent the three constraints on the number of sandwiches and drinks Mark could buy.

Can Mark buy 2 sandwiches and 2 drinks?

This situation can be represented using three inequalities. Let x be the number of sandwiches and y be the number of drinks. This system shows that the total cost must be less than or equal to \$14, and that there is at least 1 sandwich and 1 drink.

$$\begin{cases} 5x + 2y \leq 14 \\ x \geq 1 \\ y \geq 1 \end{cases}$$

Mark can buy 2 sandwiches and 2 drinks since he wants to buy at least one of each and the cost of 2 sandwiches and 2 drinks is equal to \$14, the amount of money Mark has to spend.

In some cases a number is not required for a solution. Instead, we want to know how a certain variable relates to another.

Example: Solve the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ for y_2 .

To solve for y_2 , use inverse operations. Make sure you perform the same operation on both sides of the equation.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{Write the original equation.}$$

$$m(x_2 - x_1) = y_2 - y_1 \quad \text{Multiply both sides by } x_2 - x_1$$

$$m(x_2 - x_1) + y_1 = y_2 \quad \text{Add } y_1 \text{ to both sides.}$$

REVIEW EXAMPLES

- ◆ The city of Arachna has a spider population that has been doubling every year. If there are about 100,000 spiders this year, how many will there be 4 years from now?

Solution:

Doubling means multiplying by 2, so use this equation:

$$S = 100,000 \times 2 \times 2 \times 2 \times 2 \text{ or } 100,000 \times 2^4$$

$$S = 1,600,000$$

There will be 1,600,000 spiders 4 years from now.

- ◆ The Jones family has twice as many tomato plants as pepper plants. If there are 21 plants in their garden, how many plants are pepper plants?

Solution:

A solution can be obtained by solving an equation, setting up a table, or determining the solution using a graph.

Solving an equation:

Let p = peppers and $2p$ = tomatoes.

$$p + 2p = 21$$

$$3p = 21$$

$$p = 7$$

Seven plants are pepper plants.

Setting up a table:

Let p be the number of pepper plants, and let t be the number of tomato plants. Model the situation with these two equations:

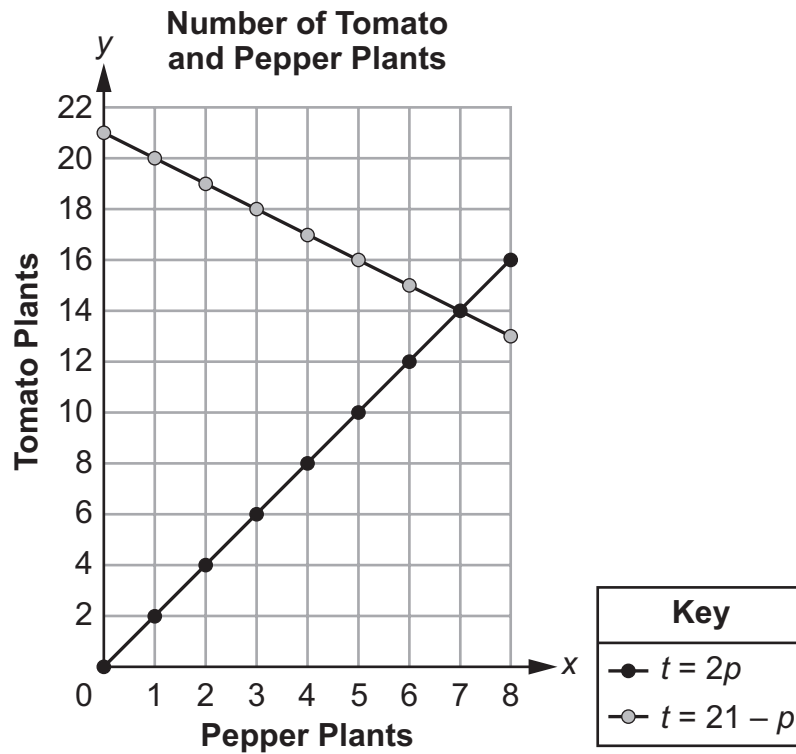
$$t = 2p$$

$$p + t = 21$$

Now create tables of values for each equation, and then graph the values. Be sure to use the same values of p for both tables. (Note: you can solve $p + t = 21$ for t to make it easier to find the values.)

$t = 2p$		$t = 21 - p$	
p	t	p	t
0	0	0	21
1	2	1	20
2	4	2	19
3	6	3	18
4	8	4	17
5	10	5	16
6	12	6	15
7	14	7	14
8	16	8	13

Using a graph:



From the graph, the equations share a common point at $(7, 14)$. That means there are 7 pepper plants and 14 tomato plants.

SAMPLE ITEMS

1. The sum of the angle measures in a triangle is 180° . Two angles of a triangle measure 20° and 50° . What is the measure of the third angle?

A. 30°
B. 70°
C. 110°
D. 160°

2. Which equation shows $P = 2l + 2w$ when solved for w ?

A. $w = \frac{2l}{P}$
B. $w = \frac{2l - P}{2}$
C. $w = 2l - \frac{P}{2}$
D. $w = \frac{P - 2l}{2}$

3. Bruce owns a business that produces widgets. He must bring in more in revenue than he pays out in costs in order to make a profit.

- It costs \$10 in labor and materials to make each of his widgets.
- His rent each month for his factory is \$4,000.

He sells each widget for \$25. What is the smallest number of widgets Bruce needs to sell each month to make a profit?

A. 160
B. 260
C. 267
D. 400

Answers to Unit 1.3 Sample Items

1. C 2. D 3. C

UNIT 2: REASONING WITH EQUATIONS AND INEQUALITIES

Unit 2 focuses on equations and inequalities. From reasoning with equations or inequalities to graphing their solutions, this unit addresses linear relationships with one or two variables. Familiarity with the properties of operations and equality is essential for mastering the skills and concepts addressed in this unit. This unit builds on your knowledge of coordinates and extends it to the use of algebraic methods to solve systems of equations.

2.1 Reasoning with Equations and Inequalities

MGSE9-12.A.REI.1 Using algebraic properties and the properties of real numbers, justify the steps of a simple, one-solution equation. Students should justify their own steps, or if given two or more steps of an equation, explain the progression from one step to the next using properties.

KEY IDEAS

Equivalent expressions produce the same result when substituting values for variables.

Example: Is the expression $\frac{6x + 8}{2}$ equivalent to $3x + 4$?

Yes, the expressions are equivalent.

$\frac{6x + 8}{2}$ Write the original expression.

$\frac{6x}{2} + \frac{8}{2}$ Use the distributive property (reversed).

$3x + 4$ Use multiplicative inverses and the identity property of 1.

To solve an equation or inequality, it is often necessary to manipulate an equation or inequality to show an equivalent form. This is often done by isolating the variable. Here is a list of what you may do to correctly solve an equation.

- Substitution: Replace a quantity, including entire expressions, with an equivalent quantity.
- Use the properties of equality or inequality.

Name of Property	Procedure	Valid for	
		Equations	Inequalities
Addition Property of Equality	Add the same positive or negative number to both sides.	✓	✓
Subtraction Property of Equality	Subtract the same positive or negative number from both sides.	✓	✓

Name of Property	Procedure	Valid for	
		Equations	Inequalities
Multiplication Property of Equality	Multiply both sides by the same positive or negative number (not 0).	✓	✓ For positive numbers only. Multiplication by a negative number reverses the inequality sign.
Division Property of Equality	Divide both sides by the same positive or negative number (not 0).	✓	✓ For positive numbers only. Division by a negative number reverses the inequality sign.

- Use the Reflexive Property of Equality and Inequality, which states for all real numbers, $x = x$. This simply means a value is equal to itself.
- Use the Transitive Property of Equality and Inequality, which states if $a = b$ and $b = c$, then $a = c$.
- Use the Symmetric Property of Equality, which states if $x = y$ then $y = x$.
- Use the Substitution Property of Equality, which states if $x = y$ then x can be substituted in for y in any equation and y can be substituted in for x in any equation.

Here are examples of when you might need to transform an equation.

Example: Solve the equation $2y + 4 = 3(2x - 6)$ for y . Show and justify your steps.

$2y + 4 = 3(2x - 6)$	Write the original equation.
$2y + 4 = 6x - 18$	Use the Distributive Property to write an equivalent expression on the right side.
$2y + 4 - 4 = 6x - 18 - 4$	Subtraction property of equality
$2y = 6x - 22$	Combine like terms on both sides, and apply additive inverse and additive identity property of 0.
$\frac{2y}{2} = \frac{6x}{2} - \frac{22}{2}$	Division property of equality
$y = 3x - 11$	Multiplicative inverses and identity property of 1



Example: Solve the equation $14 = ax + 6$ for x . Show and justify your steps.

$14 = ax + 6$	Write the original equation.
$14 - 6 = ax + 6 - 6$	Subtraction property of equality
$8 = ax$	Combine like terms on both sides, and apply additive inverse and additive identity property of 0.
$\frac{8}{a} = \frac{ax}{a}$	Division property of equality
$\frac{8}{a} = x$	Multiplicative identity property of 1

Example: Solve the inequality $4 - y > 5$ for y . Show and justify your steps.

$4 - y > 5$	Write the original inequality.
$4 - 4 - y > 5 - 4$	Subtraction property of equality
$-y > 1$	Combine like terms on both sides, and apply additive inverse and additive identity property of 0.
$\frac{-y}{-1} < \frac{1}{-1}$	Division property of equality
$y < -1$	Multiplicative inverses and identity property of 1

Important Tips

-  Know the properties of operations and the order of operations so you can readily simplify algebraic expressions and prove two expressions are equivalent.
-  Be familiar with the properties of equality and inequality. In particular, be aware that when you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality sign to preserve the relationship.

REVIEW EXAMPLES

- ◆ Are the algebraic expressions $4x - 2$ and $6x - 2(x - 1)$ equivalent?

Solution: Simplify the expression $6x - 2(x - 1)$ and see if it is equivalent to the other expression.

$$\begin{aligned} 6x - 2(x - 1) &= 6x - 2x + 2 \\ &= 4x + 2 \end{aligned}$$

No, the expressions are not equivalent.

- ◆ Solve this inequality for y : $6a - 2y > 4$

Solution:

$6a - 2y > 4$	Write the original inequality.
$6a - 6a - 2y > 4 - 6a$	Subtraction property of equality
$-2y > 4 - 6a$	Combine like terms on the left side, and apply additive inverse and additive identity property of 0.
$\frac{-2y}{-2} < \frac{4 - 6a}{-2}$	Division property of inequality and reverse the inequality symbol
$y < -2 + 3a$	Multiplicative inverses and identity property of 1

- ◆ Solve the equation $\frac{m}{6} + \frac{m}{4} = 1$ for m .

Solution:

$\frac{m}{6} + \frac{m}{4} = 1$	Write the original equation.
$2m + 3m = 12$	Multiplication property of equality
$5m = 12$	Combine like terms on the left side, and apply additive inverse and additive identity property of 0.
$\frac{5m}{5} = \frac{12}{5}$	Division property of equality
$m = \frac{12}{5}$	Multiplicative inverses and identity property of 1

- ◆ The sum of the angle measures in a triangle is 180° . The angles of a certain triangle measure x° , $2x^\circ$, and $6x^\circ$. Solve for the measure of each angle.

Solution: Use the equation $x + 2x + 6x = 180$ to represent the relationship.

$9x = 180$	Combine like terms. Apply the multiplicative identity property of 1.
$\frac{9x}{9} = \frac{180}{9}$	Division property of equality
$x = 20$	Multiplicative inverses and identity property of 1
$2x = 40$	Substitution
$6x = 120$	Substitution

The measures of the three angles are 20° , 40° , and 120° .

- ◆ A business invests \$6,000 in equipment to produce a product. Each unit of the product costs \$0.90 to produce and is sold for \$1.50. How many units of the product must be sold in order for the business to make a profit?

Solution: Let C be the total cost of producing x units. Represent the total cost with an equation.

$$C = 0.90x + 6,000$$

Let R be the total revenue from selling x units. Represent the total revenue with an equation.

$$R = 1.50x$$

A break-even point is reached when the total revenue, R , equals the total cost, C .

A profit occurs when revenue is greater than cost.

$$1.50x > 0.90x + 6,000$$

Solve the inequality for x .

Solution:

$$1.50x > 0.90x + 6,000$$

Write the original inequality.

$$0.60x > 6,000$$

Subtraction property of inequality

$$x > 10,000$$

Division property of inequality

SAMPLE ITEMS

1. Which equation shows $ax - w = 3$ solved for w ?

- A. $w = ax - 3$
- B. $w = ax + 3$
- C. $w = 3 - ax$
- D. $w = -3 - ax$

2. Which equation is equivalent to $\frac{7x}{4} - \frac{3x}{8} = 11$?

- A. $17x = 88$
- B. $11x = 88$
- C. $4x = 44$
- D. $2x = 44$

3. Which equation shows $4n = 2(t - 3)$ solved for t ?

- A. $t = \frac{4n - 2}{3}$
- B. $t = \frac{4n + 2}{3}$
- C. $t = 2n - 3$
- D. $t = 2n + 3$

4. Which equation shows $6(x + 4) = 2(y + 5)$ solved for y ?

- A. $y = x + 3$
- B. $y = x + 5$
- C. $y = 3x + 7$
- D. $y = 3x + 17$

Answers to Unit 2.1 Sample Items

1. A 2. B 3. D 4. C

2.2 Solving Equations and Inequalities

MGSE9-12.A.REI.3 Solve linear equations and inequalities in one variable including equations with coefficients represented by letters. *For example, given $ax + 3 = 7$, solve for x .*

MGSE9-12.A.REI.5 Show and explain why the elimination method works to solve a system of two-variable equations.

KEY IDEAS

Solving an equation or inequality means finding the quantity or quantities that make the equation or inequality true. The strategies for solving an equation or inequality depend on the number of variables and the exponents that are included.

Apply algebraic properties to write equivalent expressions until the desired variable is isolated on one side. Be sure to check your answers. Example: Solve $2(3 - a) = 18$.

Solve the equation using either of these two ways:

$$2(3 - a) = 18$$

$$3 - a = 9$$

$$-a = 6$$

$$a = -6$$

$$2(3 - a) = 18$$

$$6 - 2a = 18$$

$$-2a = 12$$

$$a = -6$$

Here is an algebraic method for solving a linear inequality with one variable:

Write equivalent expressions until the desired variable is isolated on one side. If you multiply or divide by a negative number, make sure you reverse the inequality symbol. Example: Solve $2(5 - x) > 8$ for x .

Solve the inequality using either of these two ways:

$$2(5 - x) > 8$$

$$5 - x > 4$$

$$-x > -1$$

$$x < 1$$




$$2(5 - x) > 8$$

$$10 - 2x > 8$$

$$-2x > -2$$

$$x < 1$$

Important Tips

-  If you multiply or divide both sides of an inequality by a negative number, make sure you reverse the inequality sign.
-  Be familiar with the properties of equality and inequality so you can transform equations or inequalities.
 - The **addition property of equality** tells us that adding the same number to each side of an equation gives us an equivalent equation.
Example: if $a - b = c$, then $a - b + b = c + b$, or $a = c + b$
 - The **multiplication property of equality** tells us that multiplying the same number to each side of an equation gives us an equivalent equation.
Example: if $\frac{a}{b} = c$, then $\frac{a}{b} \cdot b = c \cdot b$, or $a = c \cdot b$
 - The **multiplication inverse property** tells us that multiplying a number by its reciprocal equals 1.
Example: $\frac{1}{a}(a) = 1$
 - The **additive inverse property** tells us that adding a number to its opposite equals 0.
Example: $a + (-a) = 0$
-  Sometimes eliminating denominators by multiplying all terms by a common denominator or common multiple makes it easier to solve an equation or inequality.

REVIEW EXAMPLES

- ◆ Karla wants to save up for a prom dress. She figures she can save \$9 each week from the money she earns babysitting. If she plans to spend less than \$150 for the dress, how many weeks will it take her to save enough money to buy any dress in her price range?
Solution:
Let w represent the number of weeks. If she saves \$9 each week, Karla will save $9w$ dollars after w weeks. We need to determine the minimum number of weeks it will take her to save \$150. Use the inequality $9w \geq 150$ to solve the problem. We need to transform $9w \geq 150$ to isolate w . Divide both sides by 9 to get $w \geq 16\frac{2}{3}$ weeks. Because we do not know what day Karla gets paid each week, we need the answer to be a whole number. So, the answer has to be 17, the smallest whole number greater than $16\frac{2}{3}$. She will save \$144 after 16 weeks and \$153 after 17 weeks.
- ◆ Joachim wants to know if he can afford to add texting to his cell phone plan. He currently spends \$21.49 per month for his cell phone plan, and the most he can spend for his cell phone is \$30 per month. He could get unlimited text messaging added to his plan for an additional \$10 each month. Or, he could get a “pay-as-you-go” plan that charges a flat rate of \$0.15 per text message. He assumes that he will send an average of 5 text messages per day. Can Joachim afford to add a text message plan to his cell phone plan?

Solution:

Joachim cannot afford either plan.

At an additional \$10 per month for unlimited text messaging, Joachim's cell phone bill would be \$31.49 a month. If he chose the pay-as-you-go plan, each day he would expect to pay for 5 text messages. Let t stand for the number of text messages per month. Then, on the pay-as-you-go plan, Joachim could expect his cost to be represented by the expression:

$$21.49 + 0.15t$$

If he must keep his costs at \$30 or less, $21.49 + 0.15t \leq 30$.

To find the number of text messages he can afford, solve for t .

$$21.49 - 21.49 + 0.15t \leq 30 - 21.49 \quad \text{Subtract 21.49 from both sides.}$$

$$0.15t \leq 8.51 \quad \text{Combine like terms.}$$

$$t \leq 56.733 \dots \quad \text{Divide both sides by 0.15.}$$

The transformed inequality tells us that Joachim would need to send fewer than 57 text messages per month to afford the pay-as-you-go plan. However, 5 text messages per day at a minimum of 28 days in a month is 140 text messages per month. So, Joachim cannot afford text messages for a full month, and neither plan fits his budget.

- ◆ Two cars start at the same point and travel in opposite directions. The first car travels 15 miles per hour faster than the second car. In 4 hours, the cars are 300 miles apart. Use the formula below to determine the rate of the second car.

$$4(r + 15) + 4r = 300$$

What is the rate, r , of the second car?

Solution:

The second car is traveling 30 miles per hour.

$$4(r + 15) + 4r = 300 \quad \text{Write the original equation.}$$

$$4r + 60 + 4r = 300 \quad \text{Multiply 4 by } r + 15.$$

$$8r + 60 = 300 \quad \text{Combine like terms.}$$

$$8r = 240 \quad \text{Subtract 60 from each side.}$$

$$r = 30 \quad \text{Divide each side by 8.}$$

SAMPLE ITEMS

1. This equation can be used to find h , the number of hours it will take Flo and Bryan to mow their lawn.

$$\frac{h}{3} + \frac{h}{6} = 1$$

How many hours will it take them to mow their lawn?

- A. 6
- B. 3
- C. 2
- D. 1

2. A ferry boat carries passengers back and forth between two communities on the Peachville River.

- It takes 30 minutes longer for the ferry to make the trip upstream than downstream.
- The ferry's average speed in still water is 15 miles per hour.
- The river's current is usually 5 miles per hour.

This equation can be used to determine how many miles apart the two communities are.

$$\frac{m}{15 - 5} = \frac{m}{15 + 5} + 0.5$$

What is m , the distance between the two communities?

- A. 0.5 mile
- B. 5 miles
- C. 10 miles
- D. 15 miles

3. For what values of x is the inequality $\frac{2}{3} + \frac{x}{3} > 1$ true?

- A. $x < 1$
- B. $x > 1$
- C. $x < 5$
- D. $x > 5$

Answers to Unit 2.2 Sample Items

1. C 2. C 3. B

2.3 Solving a System of Two Linear Equations

MGSE9-12.A.REI.5 Show and explain why the elimination method works to solve a system of two-variable equations.

MGSE9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

KEY IDEAS

A system of linear equations consists of two or more linear equations that may or may not have a common solution. The solution of a system of two linear equations is the set of values for the variables that makes all the equations true. The solutions can be expressed as ordered pairs (x, y) or as two equations, one for x and the other for y ($x = \dots$ and $y = \dots$).

Systems of linear equations can have no solution, one solution, or many solutions. A system of linear equations will have no solution if the lines have the same slope and different y -intercepts. These lines will never intersect; therefore, there will be no value of x or y that will make the equations equal. A system of linear equations will have infinitely many solutions if the equations are scalar multiples of each other. These equations will both represent the same line; therefore, any point that lies on one of the lines will also lie on the other line. A system of linear equations will have exactly one solution if the lines intersect at exactly one point.

Strategies:

- * Use tables or graphs for solving a system of equations. For tables, use the same values for both equations.

Example: Solve this system of equations.
$$\begin{cases} y = 2x - 4 \\ x = y + 1 \end{cases}$$

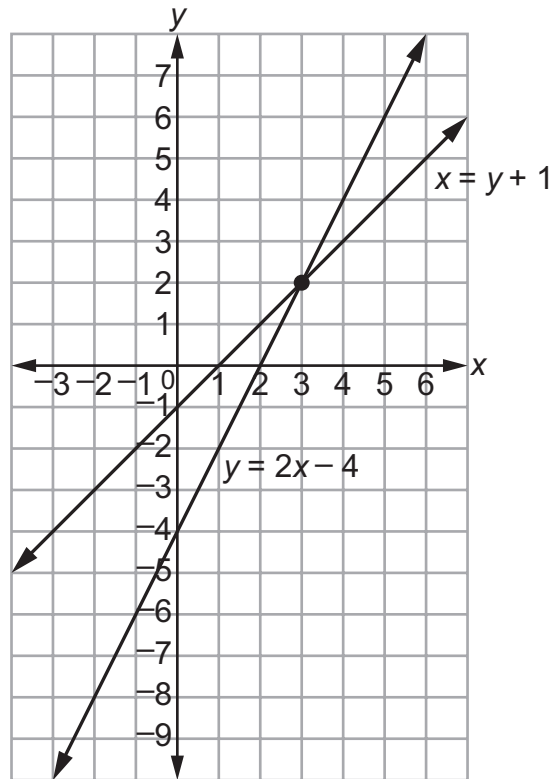
Table Method: First, find coordinates of points for each equation. Making a table of values for each is one way to do this. Use the same values for both equations. In most cases, start by using the x -values $-1, 0, 1, 2, 3$. If you are unable to find a pair of coordinates that are the same, this does not mean that there is no solution. You may need to try different values of x . This is not always the most efficient way to solve systems of equations. Notice that the second equation can be rewritten as $y = x - 1$ to solve for y in the table.

x	$2x - 4$
-1	-6
0	-4
1	-2
2	0
3	2
4	4

x	y
-1	-2
0	-1
1	0
2	1
3	2
4	3

Graph Method: For graphs, the intersection of the graphs of both equations provides the solution to the system of equations. If the lines are parallel, then there is no solution to the system. If the lines coincide, then the lines have all their points in common and any pair of points that satisfies one equation will satisfy the other.

Graph the first equation by using the y -intercept, $(0, -4)$, and the slope, 2. Graph the second equation by solving for y to get $y = x - 1$ and then use the y -intercept, $(0, -1)$, and the slope, 1. Both equations are displayed on the graph below.



The graph shows all the ordered pairs of numbers (rows from the table) that satisfy $y = 2x - 4$ and the ordered pairs that satisfy $x = y + 1$. From the graph, it appears that the lines cross at about $(3, 2)$. Try that combination in both equations to determine whether $(3, 2)$ is a solution to both equations.

$y = 2x - 4$	$x = y + 1$
$(2) = 2(3) - 4$	$3 = 2 + 1$
$2 = 6 - 4$	$3 = 3$
$2 = 2$	

So, $(3, 2)$ is the solution to the system of equations. The graph also suggests that $(3, 2)$ is the only point the lines have in common, so we have found the only pair of numbers that works for both equations.

* Simplify the problem by eliminating one of the two variables.

Substitution Method: Use one equation to isolate a variable and replace that variable in the other equation with the equivalent expression you just found. Solve for the one remaining variable. Use the solution to the remaining variable to find the unknown you eliminated.

Example: Solve this system of equations.
$$\begin{cases} 2x - y = 1 \\ 5 - 3x = 2y \end{cases}$$

Begin by choosing one of the equations and solving for one of the variables. This variable is the one you will eliminate. We could solve the top equation for y .

$$\begin{aligned}2x - y &= 1 \\2x &= 1 + y \\2x - 1 &= y \\y &= 2x - 1\end{aligned}$$

Next, use substitution to replace the variable you are eliminating in the other equation.

$$\begin{aligned}5 - 3x &= 2y \\5 - 3x &= 2(2x - 1) \\5 - 3x &= 4x - 2 \\7 &= 7x \\1 &= x\end{aligned}$$

Now, find the corresponding y -value. You can use either equation.

$$\begin{aligned}2x - y &= 1 \\2(1) - y &= 1 \\2 - y &= 1 \\-y &= 1 - 2 \\-y &= -1 \\y &= 1\end{aligned}$$

So, the solution is $x = 1$ and $y = 1$, or $(1, 1)$.

Check solution:

$$\begin{array}{ll}2x - y = 1 & 5 - 3x = 2y \\2(1) - (1) = 1 & 5 - 3(1) = 2(1) \\2 - 1 = 1 & 5 - 3 = 2 \\1 = 1 & 2 = 2\end{array}$$

Elimination Method: Add the equations (or a transformation of the equations) to eliminate a variable. Then solve for the remaining variable and use this value to find the value of the variable you eliminated.

Example: Solve this system of equations.
$$\begin{cases} 2x - y = 1 \\ 5 - 3x = -y \end{cases}$$

First, rewrite the second equation in standard form.

$$\begin{cases} 2x - y = 1 \\ -3x + y = -5 \end{cases}$$

Decide which variable to eliminate. We can eliminate the y -terms because they are opposites.

$$\begin{aligned}2x - y &= 1 \\-3x + y &= -5\end{aligned}$$

Add the equations, term by term, eliminating y and reducing to one equation. This is an application of the addition property of equality.

$$-x = -4$$

Multiply both sides by -1 to solve for x . This is an application of the multiplication property of equality.

$$\begin{aligned}(-1)(-x) &= (-1)(-4) \\ x &= 4\end{aligned}$$

Now substitute this value of x in either original equation to find y .

$$\begin{aligned}2x - y &= 1 \\ 2(4) - y &= 1 \\ 8 - y &= 1 \\ -y &= -7 \\ y &= 7\end{aligned}$$

The solution to the system of equations is $(4, 7)$.

Check solution:

$$\begin{array}{ll}2x - y = 1 & -3x + y = -5 \\ 2(4) - (7) = 1 & -3(4) + (7) = -5 \\ 8 - 7 = 1 & -12 + 7 = -5 \\ 1 = 1 & -5 = -5\end{array}$$

Example: Solve this system of equations. $\begin{cases} 3x - 2y = 7 \\ 2x - 3y = 3 \end{cases}$

First, decide which variable to eliminate. We can eliminate the y -terms but will need to change the coefficients of the y -terms so that they are the same value or opposites. We can multiply the first equation by 3 and the second equation by 2. This is the multiplication property of equality.

$$\begin{aligned}(3)(3x - 2y = 7) &\rightarrow 9x - 6y = 21 \\ (2)(2x - 3y = 3) &\rightarrow 4x - 6y = 6\end{aligned}$$

Subtract the second equation from the first equation, term by term, eliminating y and reducing to one equation. This is the addition property of equality.

$$5x = 15$$

Divide both sides by 5 to solve for x .

$$\begin{aligned}\frac{5x}{5} &= \frac{15}{5} \\ x &= 3\end{aligned}$$

Now substitute this value of x in either original equation to find y .

$$\begin{aligned}3x - 2y &= 7 \\ 3(3) - 2y &= 7 \\ 9 - 2y &= 7 \\ -2y &= -2 \\ y &= 1\end{aligned}$$

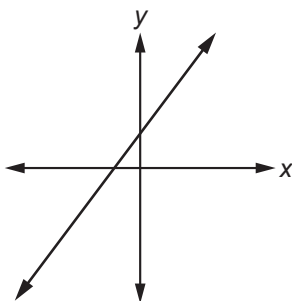
The solution to the system of equations is (3, 1).

Check solution:

$$\begin{array}{rcl} 3x - 2y & = & 7 \\ 3(3) - 2(1) & = & 7 \\ 9 - 2 & = & 7 \\ 7 & = & 7 \end{array} \qquad \begin{array}{rcl} 2x - 3y & = & 3 \\ 2(3) - 3(1) & = & 3 \\ 6 - 3 & = & 3 \\ 3 & = & 3 \end{array}$$

The graphing method only suggests the solution of a system of equations. To check the solution, substitute the values into the equations and make sure the ordered pair satisfies both equations.

When using elimination to solve a system of equations, if both variables are removed when you try to eliminate one, and if the result is a true equation such as $0 = 0$, then the lines coincide. The equations would have all ordered pairs in common, as shown in the following graph.



Example: Solve this system of equations. $\begin{cases} 3x - 3y = 3 \\ x - y = 1 \end{cases}$

First, decide which variable to eliminate. We can eliminate the y -terms but will need to change the coefficients of the y -terms so that they are the same value or opposites. We can multiply the second equation by 3. This is the multiplication property of equality.

$$(3x - 3y = 3) \rightarrow 3x - 3y = 3$$

$$(3)(x - y = 1) \rightarrow 3x - 3y = 3$$

Subtract the second equation from the first equation, term by term, eliminating y and reducing to one equation. This is the addition property of equality.

$$0 = 0$$

The solution to the system of equations is any value of x that gives the same value of y for either equation.

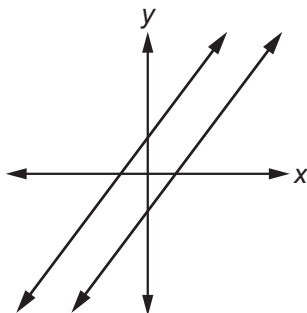
We should always check the solution to make sure it works. Because the solution works for any x , we can choose any value to check. Substitute $x = 1$ in either original equation to find y .

$$\begin{array}{rcl} 3x - 3y & = & 3 \\ 3(1) - 3y & = & 3 \\ 3 - 3y & = & 3 \\ -3y & = & 0 \\ y & = & 0 \end{array}$$

A solution to the system of equations is (1, 0). Now we plug in 1 for x and 0 for y for both equations.

$$\begin{array}{rcl} 3x - 3y = 3 & & x - y = 1 \\ 3(1) - 3(0) = 3 & & (1) - (0) = 1 \\ 3 - 0 = 3 & & 1 - 0 = 1 \\ 3 = 3 & & 1 = 1 \end{array}$$

When using elimination or substitution to solve a system of equations, if the result is a false equation such as $3 = 7$, then the lines are parallel. The system of equations has no solution since there is no point where the lines intersect.



Example: Solve this system of equations. $\begin{cases} 3x - 3y = 7 \\ x - y = 1 \end{cases}$

First, decide which variable to eliminate. We can eliminate the y -terms but will need to change the coefficients of the y -terms so that they are the same value or opposites. We can multiply the second equation by 3. This is the multiplication property of equality.

$$(3x - 3y = 7) \rightarrow 3x - 3y = 7$$

$$(3)(x - y = 1) \rightarrow 3x - 3y = 3$$

Subtract the second equation from the first equation, term by term, eliminating y and reducing to one equation. This is the addition property of equality.

$$0 = 4$$

The system of equations has no solutions.

REVIEW EXAMPLES

- ◆ Rebecca has five coins worth 65 cents in her pocket. If she only has quarters and nickels, how many quarters does she have? Use a system of equations to arrive at your answer and show all steps.

Solution:

If q represents the number of quarters and n represents the number of nickels, the two equations could be $25q + 5n = 65$ (value of quarters plus value of nickels is 65 cents) and $q + n = 5$ (she has 5 coins). The equations in the system would be $25q + 5n = 65$ and $q + n = 5$.

Next, solve $q + n = 5$ for q . By subtracting n from both sides, the result is $q = 5 - n$.

Next, eliminate q by replacing q with $5 - n$ in the other equation: $25(5 - n) + 5n = 65$.

Solve this equation for n .

$$\begin{aligned}25(5 - n) + 5n &= 65 \\125 - 25n + 5n &= 65 \\125 - 20n &= 65 \\-20n &= -60 \\n &= 3\end{aligned}$$

Now solve for q by replacing n with 3 in the equation $q = 5 - n$. So, $q = 5 - 3 = 2$, so 2 is the number of quarters.

Rebecca has 2 quarters and 3 nickels.

Check solution:

$25q + 5n = 65$	$q + n = 5$
$25(2) + 5(3) = 65$	$(2) + (3) = 5$
$50 + 15 = 65$	$2 + 3 = 5$
$65 = 65$	$5 = 5$

Note: An alternate method for finding the equations is to set up the equations in terms of dollars: $0.25q + 0.05n = 0.65$ and $q + n = 5$.

- ◆ Peg and Larry purchased “no contract” cell phones. Peg’s phone costs \$25 plus \$0.25 per minute. Larry’s phone costs \$35 plus \$0.20 per minute. After how many minutes of use will Peg’s phone cost more than Larry’s phone?

Solution:

Let x represent the number of minutes used. Peg’s phone costs $25 + 0.25x$. Larry’s phone costs $35 + 0.20x$. We want Peg’s cost to exceed Larry’s.

This gives us $25 + 0.25x > 35 + 0.20x$, which we then solve for x .

$$\begin{aligned}
 25 + 0.25x &> 35 + 0.20x \\
 25 + 0.25x - 0.20x &> 35 + 0.20x - 0.20x \\
 25 + 0.05x &> 35 \\
 25 - 25 + 0.05x &> 35 - 25 \\
 0.05x &> 10 \\
 \frac{0.05x}{0.05} &> \frac{10}{0.05} \\
 x &> 200
 \end{aligned}$$

After 200 minutes of use, Peg’s phone will cost more than Larry’s phone.

Check solution: Since Peg’s phone will cost more than Larry’s phone after 200 minutes, we can substitute 201 minutes to check if it is true.

$$\begin{aligned}
 25 + 0.25x &> 35 + 0.20x \\
 25 + 0.25(201) &> 35 + 0.20(201) \\
 25 + 50.25 &> 35 + 40.2 \\
 75.25 &> 75.20
 \end{aligned}$$

- ◆ Is $(3, -1)$ a solution of this system?

$$\begin{cases} y = 2 - x \\ 3 - 2y = 2x \end{cases}$$

Solution:

Substitute the coordinates $(3, -1)$ into each equation.

$$\begin{array}{rcl} y = 2 - x & & 3 - 2y = 2x \\ -1 = 2 - 3 & & 3 - 2(-1) = 2(3) \\ -1 = -1 & & 3 + 2 = 6 \\ & & 5 = 6 \end{array}$$

The coordinates of the given point do not satisfy $3 - 2y = 2x$. If you get a false equation when trying to solve a system algebraically, then it means that the coordinates of the point are not the solution. So, $(3, -1)$ is not a solution of the system.

- ◆ Solve this system.

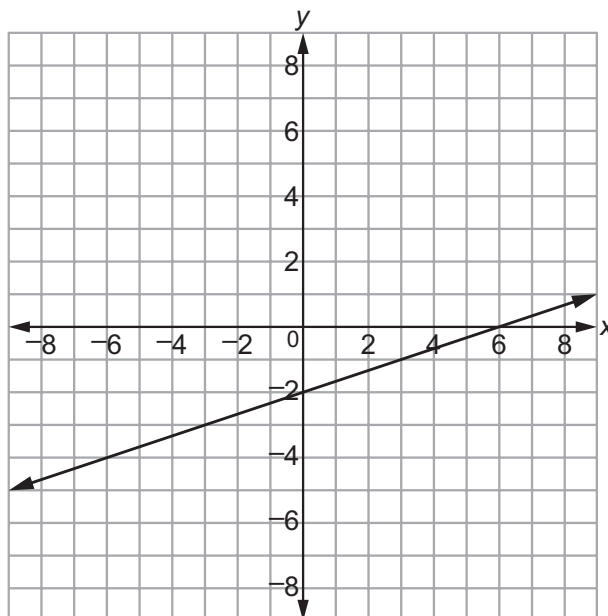
$$\begin{cases} x - 3y = 6 \\ -x + 3y = -6 \end{cases}$$

Solution:

Add the terms of the equations. Each pair of terms consists of opposites, and the result is $0 + 0 = 0$.

$$\begin{array}{r} x - 3y = 6 \\ -x + 3y = -6 \\ \hline 0 = 0 \end{array}$$

This result is always true, so the two equations represent the same line. Every point on the line is a solution to the system.

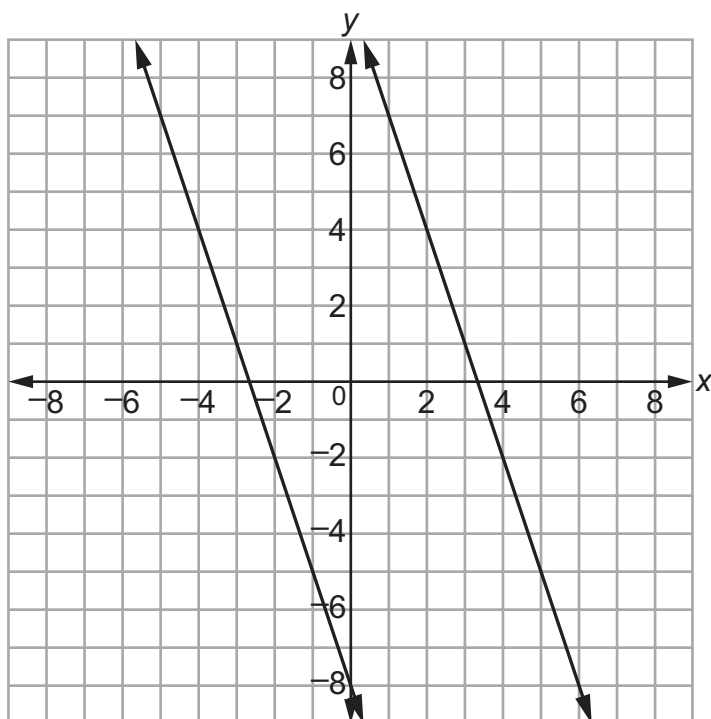


- ◆ Solve this system.
$$\begin{cases} -3x - y = 10 \\ 3x + y = -8 \end{cases}$$

Solution:

Add the terms in the equations: $0 = 2$.

The result is never true. The two equations represent parallel lines. As a result, the system has no solution.



- ◆ Look at the tables of values for two linear functions, $f(x)$ and $g(x)$.

x	$f(x)$	x	$g(x)$
-1	16	-1	-18
0	7	0	-14
1	4	1	-10
3	-2	3	-2
5	-8	5	6
7	-14	7	14

What is the solution to $f(x) = g(x)$?

Solution:

The solution to $f(x) = g(x)$ is $x = 3$. This is the value of x where $f(x)$ and $g(x)$ both equal -2 .

SAMPLE ITEMS

1. Which ordered pair is a solution of $3y + 2 = 2x - 5$?

- A. $(-5, 2)$**
- B. $(0, -5)$**
- C. $(5, 1)$**
- D. $(7, 5)$**

2. A manager is comparing the cost of buying baseball caps from two different companies.

- Company X charges a \$50 fee plus \$7 per baseball cap.**
- Company Y charges a \$30 fee plus \$9 per baseball cap.**

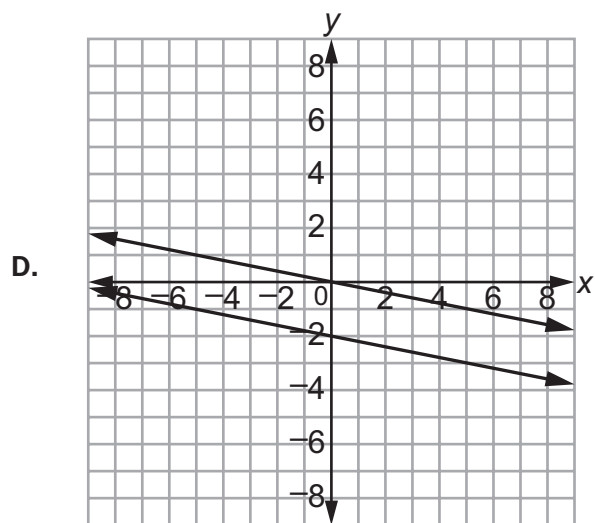
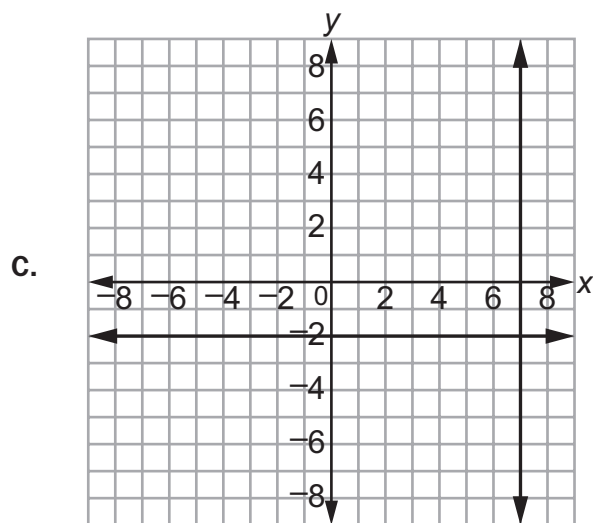
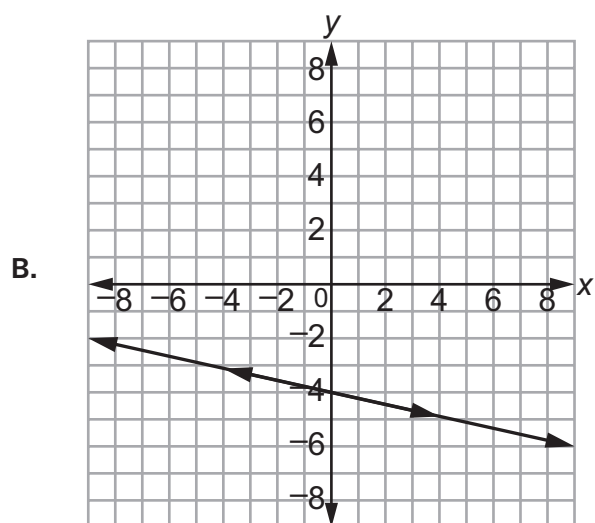
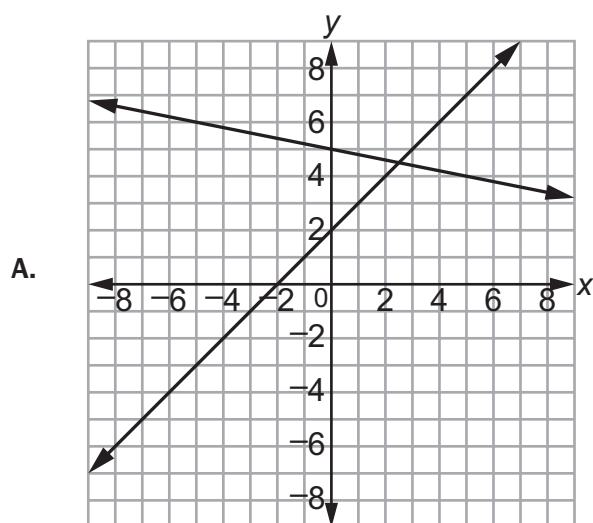
For what number of baseball caps will the cost be the same at both companies?

- A. 10**
- B. 20**
- C. 40**
- D. 100**

3. A shop sells one-pound bags of peanuts for \$2 and three-pound bags of peanuts for \$5. If 9 bags are purchased for a total cost of \$36, how many three-pound bags were purchased?

- A. 3**
- B. 6**
- C. 9**
- D. 18**

4. Which graph represents a system of linear equations that has multiple common coordinate pairs?



Answers to Unit 2.3 Sample Items

1. C 2. A 3. B 4. B

2.4 Graphing the Solutions of Equations and Inequalities

MGSE9-12.A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

MGSE9-12.A.REI.12 Graph the solution set to a linear inequality in two variables.

KEY IDEAS

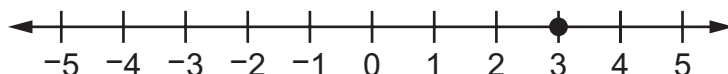
The solution to an equation or inequality can be displayed on a graph using a coordinate or coordinates. If the equation or inequality involves only one variable, then the number line is one coordinate system that can be used.

Example: Use a number line to display the solution to $3x + 5 = 14$.

First, solve the equation.

$$\begin{aligned}3x + 5 &= 14 \\3x + 5 - 5 &= 14 - 5 \\3x &= 9 \\x &= 3\end{aligned}$$

After solving the equation, we get $x = 3$. Since there is only one variable, a number line can be used to display the answer.



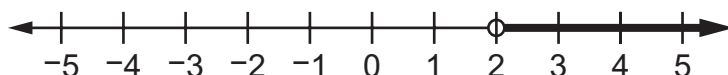
For an equation, the display shows the value(s) on the number line that satisfy the equation. Typically, a dot is placed on the line where the solution lies.

Example: Use a number line to display the solution to $3x + 8 > 14$.

First, solve the inequality.

$$\begin{aligned}3x + 8 &> 14 \\3x + 8 - 8 &> 14 - 8 \\3x &> 6 \\x &> 2\end{aligned}$$

After solving the inequality through the application of algebraic properties, the result is $x > 2$. Again, use a number line to display the answer. For an inequality, the display shows the values on the number line that satisfy the inequality. The display is usually a ray drawn on the number line that may or may not include its starting point. The inequality uses the $>$ symbol, so use an open circle to show that 2 is not a solution.



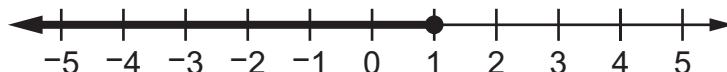
The solution is all values of x greater than 2.

Example: Use a number line to display the solution to $7 - 4x \geq 3$.

First, solve the inequality.

$$\begin{aligned} 7 - 4x &\geq 3 \\ 7 - 4x - 7 &\geq 3 - 7 \\ -4x &\geq -4 \\ x &\leq 1 \end{aligned}$$

The solution of the inequality is $x \leq 1$. Because the inequality uses the \leq symbol, use a closed circle to show that 1 is a solution.



The solution is all values of x less than or equal to 1.

If the equation or inequality involves two variables, then a coordinate plane is used to display the solution. For an equation with two variables, the display shows the points (ordered pairs) that satisfy the equation. The display should look like a curve, line, or part of a line, depending on the situation.

Example: Use a Cartesian plane to display the solution to $3x + y = 14$.

First, solve the equation for y .

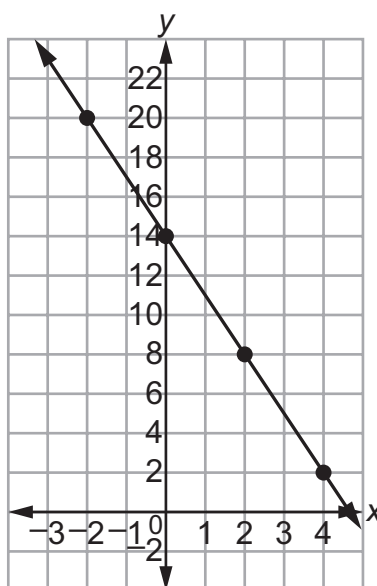
$$\begin{aligned} 3x + y &= 14 \\ 3x - 3x + y &= 14 - 3x \\ y &= 14 - 3x \end{aligned}$$

We will need to determine some ordered pairs of numbers for x and y that satisfy the equation. A good way to do this is to organize your findings in an input/output table with a column for x and a column for y , as shown below.

x	$14 - 3x$	y
-2	$14 - 3(-2)$	20
0	$14 - 3(0)$	14
2	$14 - 3(2)$	8
4	$14 - 3(4)$	2

Use the numbers in the first and last columns of the table as the coordinates of the points on your graph. Since the function is continuous, connect the points. If the points were the only numbers you could use, then you would not have to connect the points.

Here is the graph of $3x + y = 14$.



For an inequality, the graph of the solutions is shown by a shaded half-plane. Points that lie in the shaded area are solutions to the inequality. If values on the boundary line are not solutions ($<$, $>$), then the line is dashed. If the values on the boundary line are solutions (\leq , \geq), then the line is solid.

Example: Use a Cartesian plane to display the solution to $3x + y > -1$.

First, solve the inequality for y .

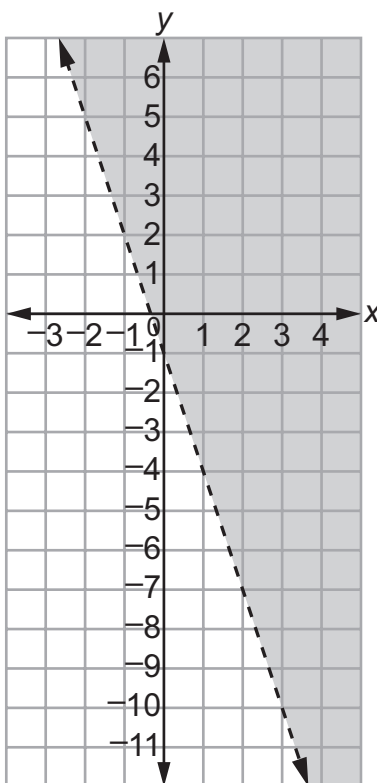
$$\begin{aligned} 3x + y &> -1 \\ 3x - 3x + y &> -1 - 3x \\ y &> -1 - 3x \quad \text{or} \quad y > -3x - 1 \end{aligned}$$

We will need to determine some ordered pairs of numbers for x and y that fit the boundary line $y = -3x - 1$.

You can graph the line using the slope and y -intercept of the equation. The slope is -3 and the y -intercept is -1 . Using the y -intercept, plot the point at $(0, -1)$. From the point, use the slope to plot another point with a value of y that is three less than -1 and one more than 0 . This gives a point at $(1, -4)$. Next, since the inequality used the $>$ symbol, use a dashed line through the two points.

Next, decide which side of the boundary line to shade. Choose a test point not on the line. If this point is a solution to the inequality, shade the region that includes this point. If the point is not a solution, shade the region that does not include this point. It is usually easy to use $(0, 0)$ as a test point when it is not on the line.

Is $3(0) + 0 > -1$? Yes, so $(0, 0)$ is a solution of the inequality. Shade the region above the line. The graph for $3x + y > -1$ is shown.



Example: Graph the solutions of $y + 2 \leq x$.

First, solve the inequality for y .

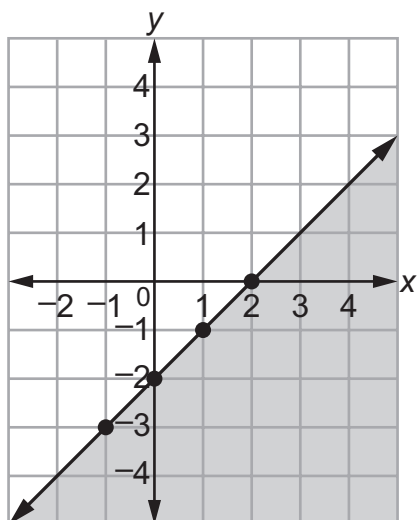
$$\begin{aligned} y + 2 &\leq x \\ y + 2 - 2 &\leq x - 2 \\ y &\leq x - 2 \end{aligned}$$

Create a table of values for the boundary line $y = x - 2$, or graph the line using the slope and y -intercept.

x	$x - 2$	y
-1	$(-1) - 2$	-3
0	$(0) - 2$	-2
1	$(1) - 2$	-1
2	$(2) - 2$	0

Use the points to draw the boundary line. Use a solid line since the inequality uses the \leq symbol.

Again, use $(0, 0)$ as a test point. Is $0 + 2 \leq 0$? No, so $(0, 0)$ is not a solution of the inequality. Shade the region below the line. The graph for $y + 2 \leq x$ is represented below.



REVIEW EXAMPLE

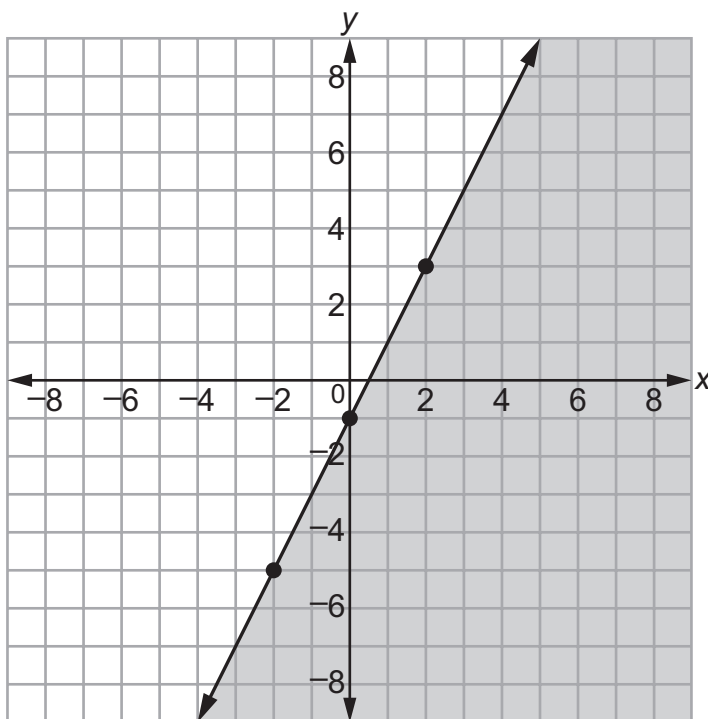
- ◆ Graph the solution region for $y \leq 2x - 1$.

Solution:

First, graph the line using the slope and y-intercept of the equation. The slope is 2 and the y-intercept is -1 . Using the y-intercept, plot the point at $(0, -1)$. From the point, use the slope to plot another point with a value of y that is two greater than -1 and one more than 0. This gives a point at $(1, 1)$. Since the inequality used the \leq symbol, use a solid line through the two points.

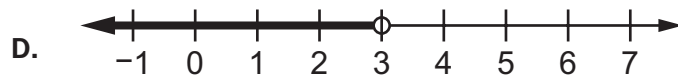
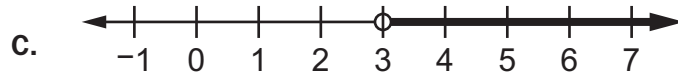
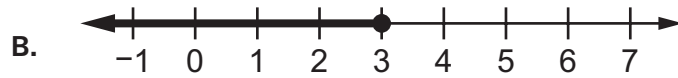
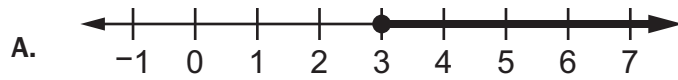
Next, decide which side of the boundary line to shade. Use $(0, 0)$ as a test point.

Is $0 \leq 2(0) - 1$? No, so $(0, 0)$ is not a solution of the inequality. Shade the region below the line. The graph for $y \leq 2x - 1$ is represented below.

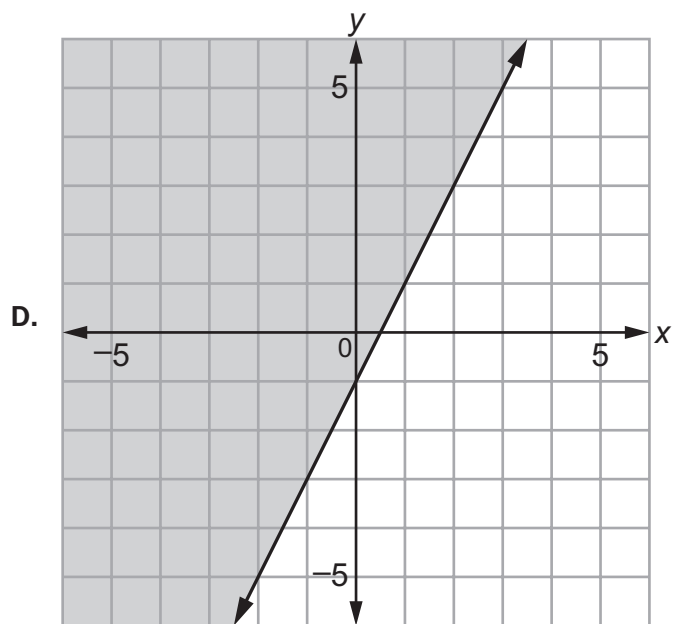
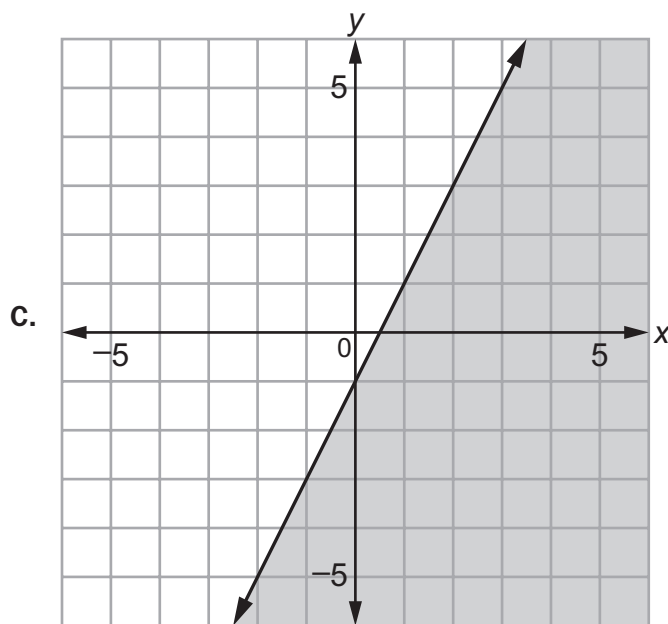
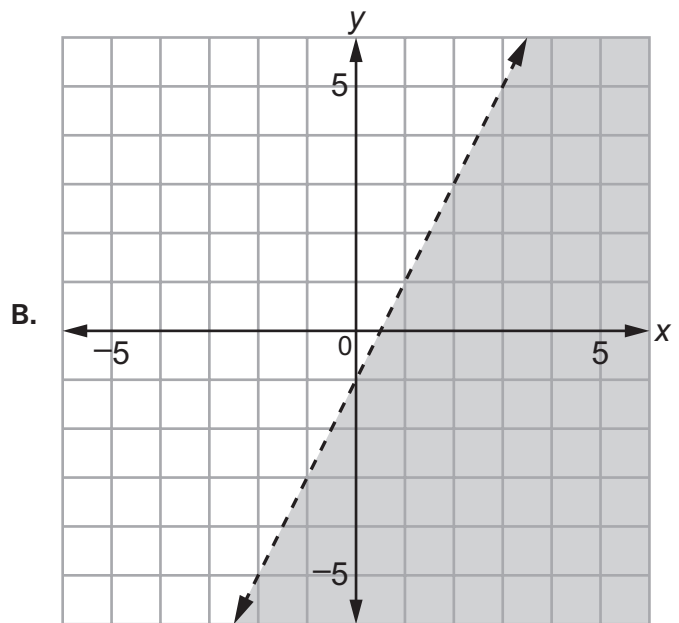
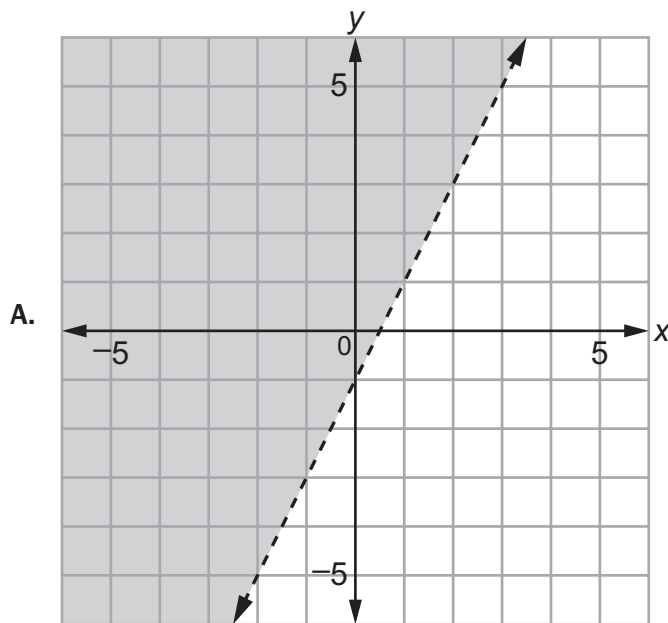


SAMPLE ITEMS

1. Which graph represents $x > 3$?



2. Which graph represents $y > 2x - 1$?



Answers to Unit 2.4 Sample Items

1. C 2. A

UNIT 3: LINEAR AND EXPONENTIAL FUNCTIONS

In Unit 3, students will learn function notation and develop the concepts of domain and range. Students will explore different ways of representing functions (e.g., graphs, rules, tables, and sequences) and interpret functions given graphically, numerically, symbolically, and verbally. Discovering how functions can be transformed, similar to shapes in geometry, and learning about how parameters affect functions are aspects of this unit. Students will also learn how to compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They will interpret arithmetic sequences as linear functions and geometric sequences as exponential functions.

3.1 Represent and Solve Equations and Inequalities Graphically

MGSE9-12.A.REI.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane.

MGSE9-12.A.REI.11 Using graphs, tables, or successive approximations, show that the solution to the equation $f(x) = g(x)$ is the x -value where the y -values of $f(x)$ and $g(x)$ are the same.

KEY IDEAS

The graph of a linear equation in two variables is a collection of ordered pair solutions in a coordinate plane. It is a graph of a straight line. Often tables of values are used to organize the ordered pairs.

Example: Every year, Silas buys fudge at the state fair. He buys two types: peanut butter and chocolate. This year he intends to buy \$24 worth of fudge. If chocolate costs \$4 per pound and peanut butter costs \$3 per pound, what are the different combinations of fudge that he can purchase if he only buys whole pounds of fudge?

If we let x be the number of pounds of chocolate and y be the number of pounds of peanut butter, we can use the equation $4x + 3y = 24$. Now we can solve this equation for y to make it easier to complete our table.

$$4x + 3y = 24$$

Write the original equation.

$$4x - 4x + 3y = 24 - 4x$$

Addition Property of Equality

$$3y = 24 - 4x$$

Additive Inverse Property

$$\frac{3y}{3} = \frac{24 - 4x}{3}$$

Multiplication Property of Equality

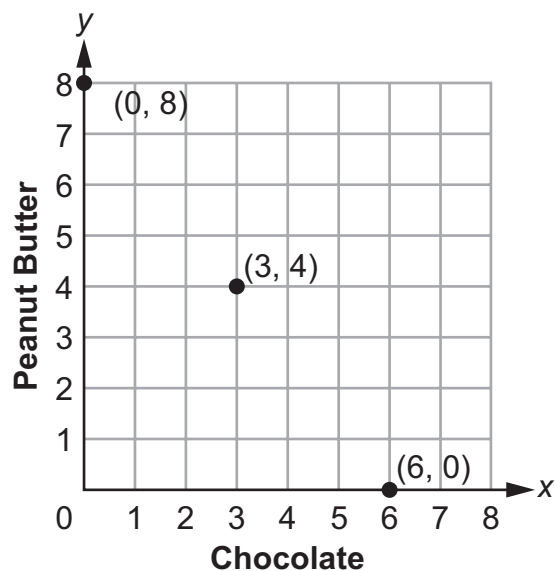
$$y = \frac{24 - 4x}{3}$$

Multiplicative Inverse Property

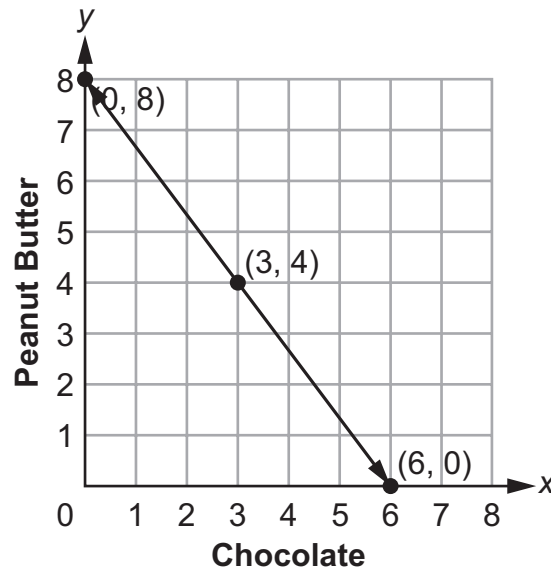
We will only use whole numbers in the table because Silas will only buy whole pounds of fudge.

Chocolate (x)	Peanut butter (y)
0	8
1	$6\frac{2}{3}$ (not a whole number)
2	$5\frac{1}{3}$ (not a whole number)
3	4
4	$2\frac{2}{3}$ (not a whole number)
5	$1\frac{1}{3}$ (not a whole number)
6	0

The ordered pairs from the table that we want to use are (0, 8), (3, 4), and (6, 0). The graph would look like the one shown:



Based on the number of points in the graph, there are three possible ways that Silas can buy pounds of fudge: 8 pounds of peanut butter only, 3 pounds of chocolate and 4 pounds of peanut butter, or 6 pounds of chocolate only. Notice that if the points on the graph were joined, they would form a line. If Silas allowed himself to buy partial pounds of fudge, then there would be many more possible combinations. Each combination would total \$24 and be represented by a point on the line that contains $(0, 8)$, $(3, 4)$, and $(6, 0)$. Note, though, that negative amounts of chocolate or peanut butter are not possible, so we do not consider any negative values for this line.



REVIEW EXAMPLES

◆ Consider the equations $y = 2x - 3$ and $y = -x + 6$.

a. Complete the tables below.

$y = 2x - 3$	
x	y
-1	
0	
1	
2	
3	

$y = -x + 6$	
x	y
-1	
0	
1	
2	
3	

- b. Is there an ordered pair that satisfies both equations? If so, what is it?
- c. Graph both equations on the same coordinate plane by plotting the ordered pairs from the tables and connecting the points.
- d. Do the lines appear to intersect? If so, where? How can you tell that the point where the lines appear to intersect is a common point for both lines?

Solution:

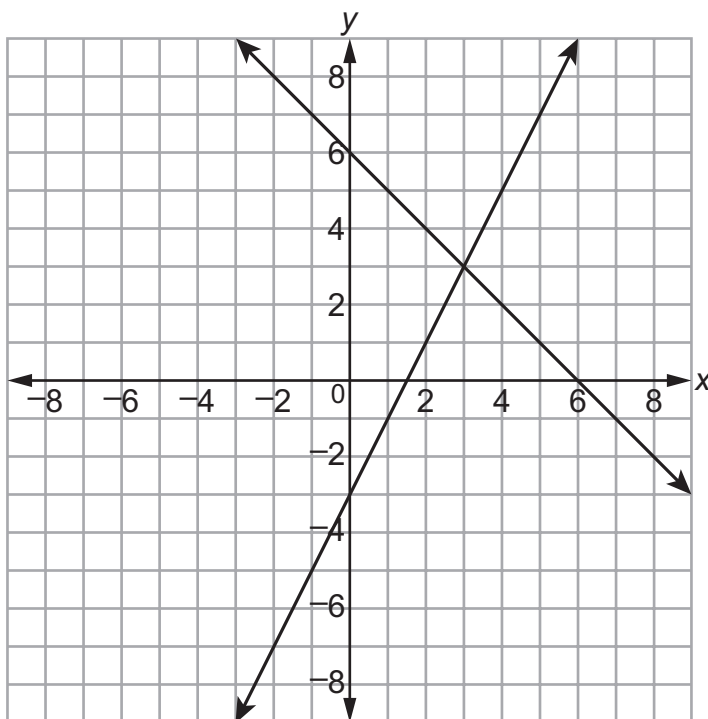
a.

$y = 2x - 3$	
x	y
-1	-5
0	-3
1	-1
2	1
3	3

$y = -x + 6$	
x	y
-1	7
0	6
1	5
2	4
3	3

- b. Yes, the ordered pair (3, 3) satisfies both equations.

c.



- d. The lines appear to intersect at $(3, 3)$. When $x = 3$ and $y = 3$ are substituted into each equation, the values satisfy both equations. This proves that $(3, 3)$ lies on both lines, which means it is a common solution to both equations.

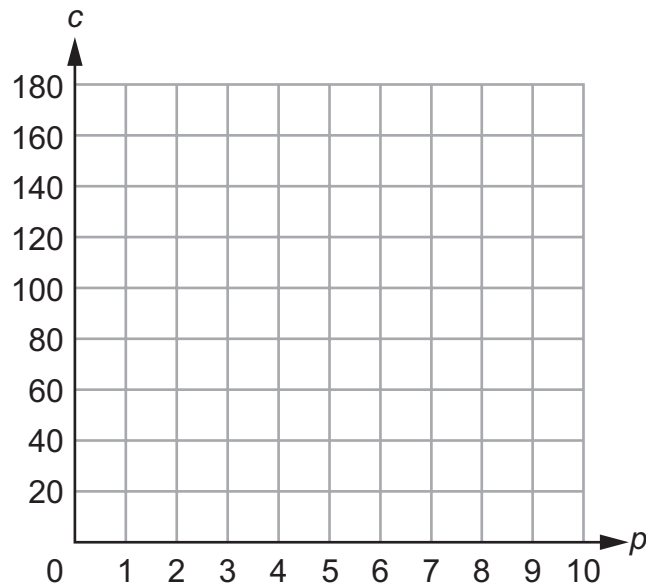
- ♦ Elliot is planning a trip with his cousins and is deciding between two different cabin rentals.

Cozy Cabins	Cabana Cabins
\$12 per person	\$9 per person
\$56 cleaning fee	\$74 cleaning fee

He writes the following system of equations to model the cost of the cabin rentals. In this system, c represents the cost of a cabin and p represents the number of people staying in the cabin.

$$\begin{cases} c = 12p + 56 \\ c = 9p + 74 \end{cases}$$

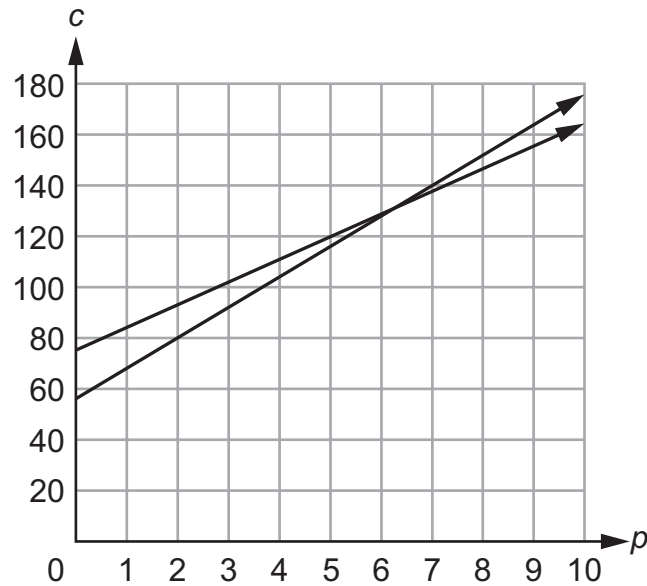
- a. Graph the system of equations for 0 to 10 people.



- b. If Elliot and two of his cousins go on the trip, which cabin rental should they use? Justify your reasoning.
- c. For what number of people is the cost at Cozy Cabins the same as the cost at Cabana Cabins? Explain how you found your answer.
- d. Explain how Elliot can use the answer from c to decide which cabin rental is less expensive for various numbers of people.

Solution:

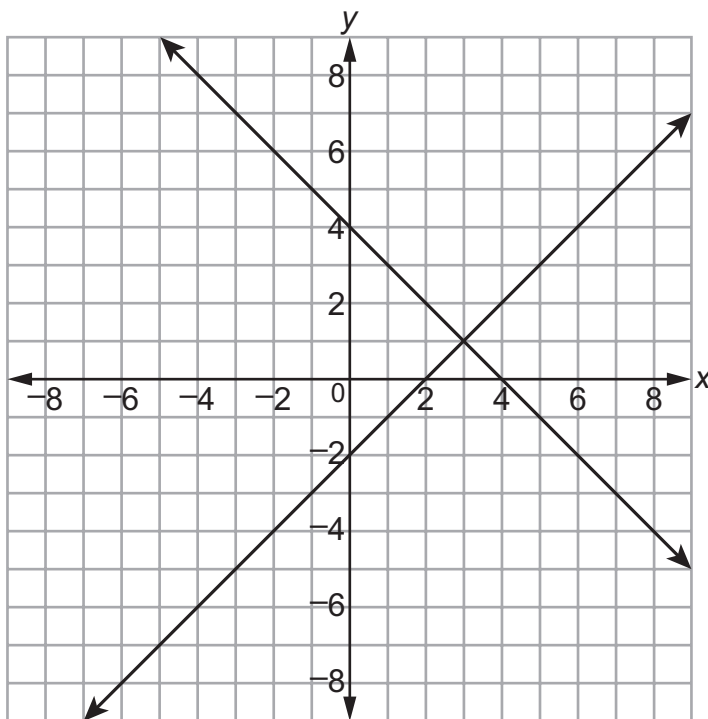
a.



- b. Elliot and his cousins should use Cozy Cabins. For three people Cozy Cabins costs \$92 and at Cabana Cabins the cost is \$101.
- c. For 6 people, the cost is the same at both cabin rentals. In the graph, it is where the two lines cross. I substituted 6 for p into both equations and they had the same cost.
- d. Cozy Cabins is less expensive for 1 to 5 people. Cabana Cabins is less expensive for any number of people 7 or greater.

SAMPLE ITEMS

1. Two lines are graphed on this coordinate plane.



Which point appears to be a solution of the equations of both lines?

- A. $(0, -2)$
- B. $(0, 4)$
- C. $(2, 0)$
- D. $(3, 1)$

2. Based on the tables, at what point do the lines $y = -x + 5$ and $y = 2x - 1$ intersect?

$y = -x + 5$		$y = 2x - 1$	
x	y	x	y
-1	6	-1	-3
0	5	0	-1
1	4	1	1
2	3	2	3
3	2	3	5

- A. (1, 1)
- B. (3, 5)
- C. (2, 3)
- D. (3, 2)

Answers to Unit 3.1 Sample Items

1. D 2. C

3.2 Understand the Concept of a Function and Use Function Notation

MGSE9-12.F.IF.1 Understand that a function from one set (the input, called the domain) to another set (the output, called the range) assigns to each element of the domain exactly one element of the range, i.e. each input value maps to exactly one output value. If f is a function, x is the input (an element of its domain), and $f(x)$ is the output (an element of the range). Graphically, the graph is $y = f(x)$.

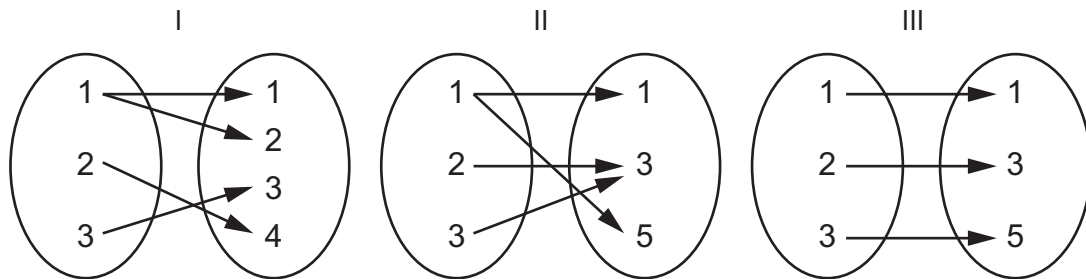
MGSE9-12.F.IF.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

MGSE9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. (Generally, the scope of high school math defines this subset as the set of natural numbers 1, 2, 3, 4, ...) By graphing or calculating terms, students should be able to show how the recursive sequence $a_1 = 7$, $a_n = a_{n-1} + 2$; the sequence $s_n = 2(n - 1) + 7$; and the function $f(x) = 2x + 5$ (when x is a natural number) all define the same sequence.

KEY IDEAS

There are many ways to show how pairs of quantities are related. Some of them are shown below.

- Mapping Diagrams**



- Sets of Ordered Pairs**

Set I: $\{(1, 1), (1, 2), (2, 4), (3, 3)\}$

Set II: $\{(1, 1), (1, 5), (2, 3), (3, 3)\}$

Set III: $\{(1, 1), (2, 3), (3, 5)\}$

- Tables of Values**

I

x	y
1	1
1	2
2	4
3	3

II

x	y
1	1
1	5
2	3
3	3

III

x	y
1	1
2	3
3	5

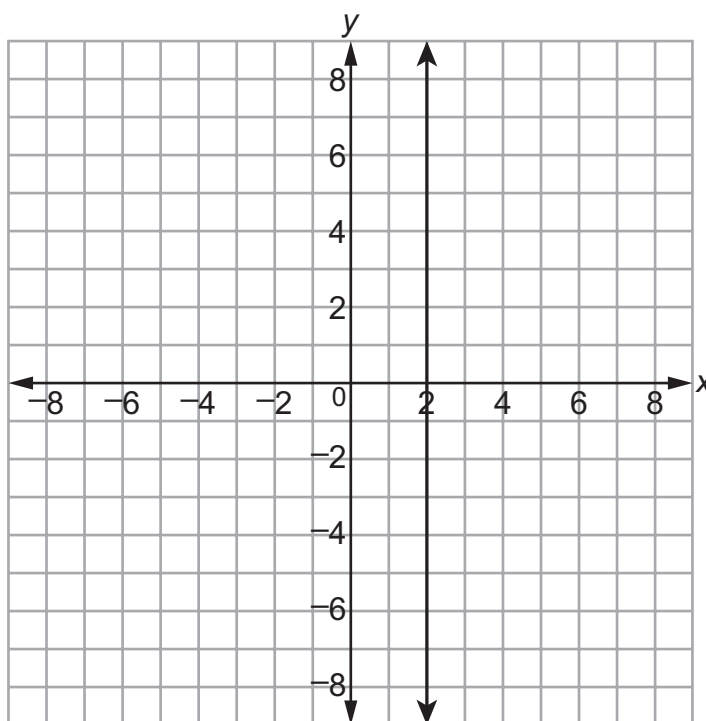
The relationship shown in Mapping Diagram I, Set I, and Table I all represent the same paired numbers. Likewise, Mapping Diagram II, Set II, and Table II all represent the same quantities. The same goes for the third group of displays.

Notice the arrows in the mapping diagrams are all arranged from left to right. The numbers on the left side of the mapping diagrams are the same as the x -coordinates in the ordered pairs as well as the values in the first columns of the tables. Those numbers are called the input values of a quantitative relationship and are known as the **domain**. The numbers on the right of the mapping diagrams, the y -coordinates in the ordered pairs, and the values in the second columns of the tables are the output values, or **range**. Every number in the domain is assigned to at least one number of the range.

Mapping diagrams, ordered pairs, and tables of values are good to use when there are a limited number of input and output values. There are some instances when the domain has an infinite number of elements to be assigned. In those cases, it is better to use either an algebraic rule or a graph to show how pairs of values are related. Often we use equations as the algebraic rules for the relationships. The domain can be represented by the independent variable and the range can be represented by the dependent variable.

A **function** is a quantitative relationship where each member of the domain is assigned to exactly one member of the range. Of the relationships in the tables of values, only III is a function. In I and II, there were members of the domain that were assigned to two elements of the range. In particular, in I, the value 1 of the domain was paired with 1 and 2 of the range. The relationship is a function if two values in the domain are related to the same value in the range.

Consider the vertical line $x = 2$. Every point on the line has the same x -value and a different y -value. So the value of the domain is paired with infinitely many values of the range. This line is not a function. In fact, all vertical lines are not functions.



A function can be described using a **function rule** that represents an output value, or element of the range, in terms of an input value, or element of the domain.

A function rule can be written in **function notation**. Here is an example of a function rule and its notation.

$$y = 3x + 5$$

y is the output and x is the input.

$$f(x) = 3x + 5$$

Read as “ f of x .”

$$f(2) = 3(2) + 5$$

“ f of 2,” the value of the function at $x = 2$, is the output when 2 is the input.

Be careful—do not confuse the parentheses used in notation with multiplication.

Functions can also represent real-life situations; for example, $f(15) = 45$ can represent 15 books that cost \$45. Functions can have restrictions or constraints, such as only including whole numbers, as in the situation of the number of people in a class and the number of books in the class. There cannot be half a person or half a book.

Note that all functions have a corresponding graph. The points that lie on the graph of a function are formed using input values, or elements of the domain, as the x -coordinates and output values, or elements of the range, as the y -coordinates.

Example: Given $f(x) = 2x - 1$, find $f(7)$.

$$f(7) = 2(7) - 1 = 14 - 1 = 13$$

Example: If $g(6) = 3 - 5(6)$, what is $g(x)$?

$$g(x) = 3 - 5x$$

Example: If $f(-2) = -4(-2)$, what is $f(b)$?

$$f(b) = -4b$$

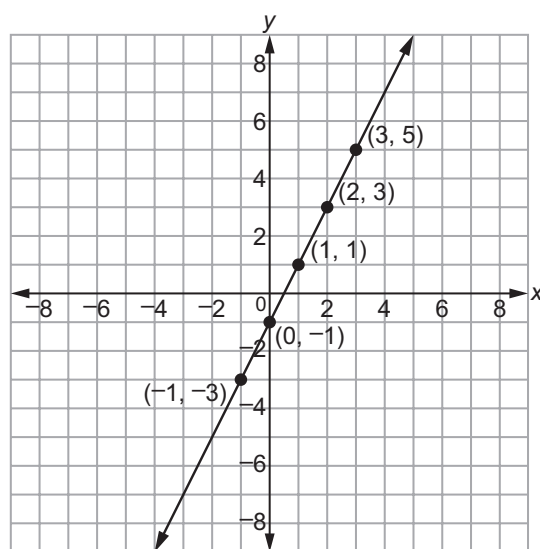
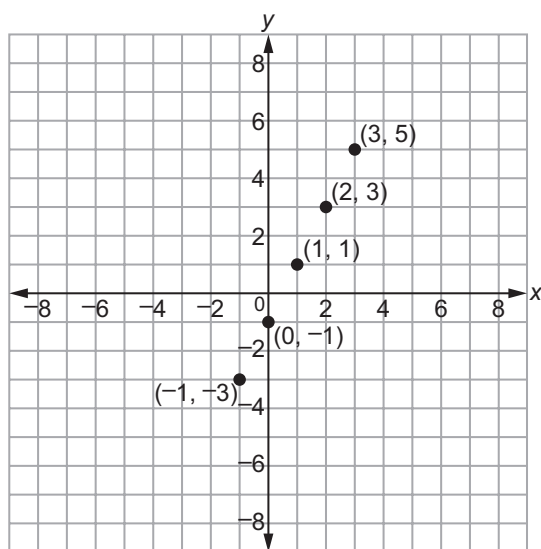
Example: Graph $f(x) = 2x - 1$.

In the function rule $f(x) = 2x - 1$, $f(x)$ is the same as y .

Then we can make a table of x (input) and y (output) values.

x	$f(x)$
-1	-3
0	-1
1	1
2	3
3	5

The values in the rows of the table form ordered pairs. We plot those ordered pairs. If the domain is not specified, we connect the points. If the numbers in the domain are not specified, we assume that they are all real numbers. If the domain is specified such as whole numbers only, then connecting the points is not needed.



A **sequence** is an ordered list of numbers. Each number in the sequence is called a **term**. The terms are consecutive, or identified as the first term, second term, third term, and so on. The pattern in the sequence is revealed in the relationship between each term and its term number or in a term's relationship to the previous term in the sequence.

Example: Consider the sequence 3, 6, 9, 12, 15, . . . The first term is 3, the second term is 6, the third term is 9, and so on. The ". . ." at the end of the sequence indicates the pattern continues without end. Can this pattern be considered a function?

There are different ways of seeing a pattern in the sequence. The initial term (y-intercept) and the slope can be used to create a table to derive the function. One way is to say each number in the sequence is 3 times the number of its term. For example, the fourth term would be 3 times 4, or 12. Looking at the pattern in this way, all you would need to know is the number of the term, and you could predict the value of the term. The value of each term would be a function of its term number. We could use this relationship to write an algebraic rule for the sequence, $y = 3x$, where x is the number of the term and y is the value of the term. This algebraic rule would assign only one number to each input value from the numbers 1, 2, 3, etc., so we could write a function for the sequence. We can call the function T and write its rule as $T(n) = 3n$, where n is the term number and 3 is the difference between each term in the sequence, called the common difference. The domain for the function T would be counting numbers. The range would be the value of the terms in the sequence. When an equation with the term number as a variable is used to describe a sequence, we refer to it as the **explicit formula** for the sequence, or the **closed form**. We could also use the common difference and the initial term to find the explicit formula by using $a_n = a_1 + d(n - 1)$, where a_1 is the first term and d is the common difference. We can create the explicit function $T(n) = 3(n - 1) + 3$ for all $n > 0$. The domain for this function would be natural numbers.

Another way to describe the sequence in the example is to say each term is three more than the term before it. Instead of using the number of the term, you would need to know a previous term to find a subsequent term's value. We refer to a sequence represented in this form as a **recursive formula**.

If the n th term of a sequence and the common difference between consecutive terms is known, you can find the $(n + 1)$ th term using the **recursive formula** $a_n = a_{n-1} + d$, where a_n is the n th term, n is the number of a term, $n - 1$ is the number of the previous term, and d is the common difference.

Take the sequence 3, 6, 9, 12, 15, . . . as an example. We can find the sixth term of the sequence using the fourth and fifth terms.

The common difference d is $15 - 12 = 3$. So the sixth term is given by $a_6 = a_5 + 3$. $a_6 = 15 + 3 = 18$.

Example: Consider this sequence: 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$. One way to look at this pattern is to say each successive term is half the term before it, and the first term is 16. With this approach you could easily determine the terms for a limited or **finite** sequence. Another way would be to notice that each term is 32 times a power of $\frac{1}{2}$. If n represents the number of the term, each term is 32 times $\frac{1}{2}$ raised to the n th power, or $32 \cdot \left(\frac{1}{2}\right)^n$. This approach lends itself to finding an explicit formula for any missing term if you know its term number. That is, the value of each term would depend on the term number. Using the initial terms as 32, the domain is the set of natural numbers. If we are to use the first term (16) in $16 \cdot \left(\frac{1}{2}\right)^n$, the domain would be counting numbers. The value $\frac{1}{2}$ is multiplied by a term to get the following term. This is called the **common ratio**.

Also, the patterns in sequences can be shown by using tables. For example, this table shows the above sequence:

Term Number (n)	1	2	3	4	5	6	7	8
Value (a_n)	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Notice the numbers in the top row of the table are consecutive counting numbers starting with one and increasing to the right. The sequence has eight terms, with 16 being the value of the first term and $\frac{1}{8}$ being the value of the eighth term. A sequence with a specific number of terms is finite. If a sequence continues indefinitely, it is called an **infinite sequence**.

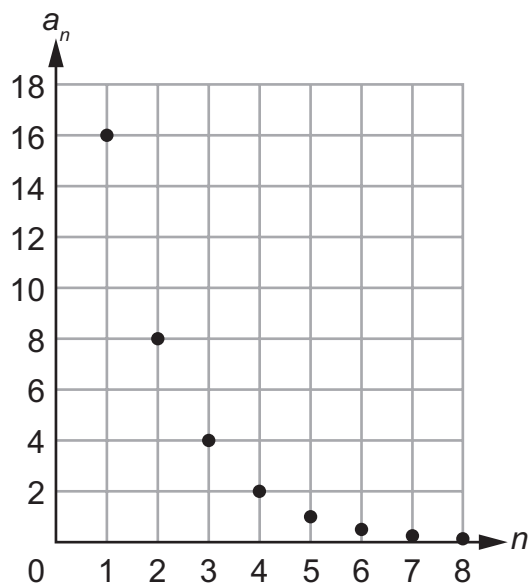
If the value of the first term of a sequence and the common ratio between consecutive terms are known, you can find the value of the n th term using the **recursive formula** $a_n = a_{n-1} \cdot r$, where a_n is the value of the n th term, a_1 is the known value of the first term, n is the term number, $n - 1$ is the previous term, and r is the common ratio.

Take the sequence 16, 8, 4, 2, 1, . . . as an example. We can find the value of the sixth term of the sequence using the recursive formula.



The common ratio r is $\frac{1}{2}$ and the fifth term, a_5 , is 1. So the sixth term is given by

$$a_6 = a_5 \cdot \left(\frac{1}{2}\right) = 1 \cdot \left(\frac{1}{2}\right) = \frac{1}{2}.$$

Also, notice the graph of the sequence. The points are not connected with a curve because the sequence is discrete and not continuous.



Important Tips

-  Use language carefully when talking about functions. For example, use f to refer to the function as a whole and use $f(x)$ to refer to the output when the input is x .
-  The advantage of using an explicit formula over a recursive formula is to quickly determine the value of the n th term of the function. However, a recursive formula helps you see the pattern occurring between sequential terms. Not all sequences can be represented as functions. Be sure to check all the terms you are provided with before reaching the conclusion that there is a pattern. For example, consider the sequence: 0, 4, 2, 10, 4, 12, -16, 5. This sequence cannot be represented by a function since the second term and the fifth term are the same.

REVIEW EXAMPLES

- ◆ A manufacturer keeps track of her monthly costs by using a “cost function” that assigns a total cost for a given number of manufactured items, x . The function is $C(x) = 5,000 + 1.3x$.
- What is the reasonable domain of the function?
 - What is the cost of 2,000 items?
 - If costs must be kept below \$10,000 this month, what is the greatest number of items she can manufacture?

Solution:

- Since x represents a number of manufactured items, it cannot be negative, nor can a fraction of an item be manufactured. Therefore, the domain can only include values that are whole numbers.
- Substitute 2,000 for x : $C(2,000) = 5,000 + 1.3(2,000) = \$7,600$
- Form an inequality:

$$\begin{aligned} C(x) &< 10,000 \\ 5,000 + 1.3x &< 10,000 \\ 1.3x &< 5,000 \\ x &< 3,846.2, \text{ or } 3,846 \text{ items} \end{aligned}$$

- ◆ Consider the first six terms of this sequence: 1, 3, 9, 27, 81, 243, . . .
- What is a_1 ? What is a_3 ?
 - What is the reasonable domain of the function?
 - If the sequence defines a function, what is the range?
 - What is the common ratio of the function?

Solution:

- a_1 is 1 and a_3 is 9.
- The domain is the set of counting numbers: $\{1, 2, 3, 4, 5, \dots\}$.
- The range is: $\{1, 3, 9, 27, 81, 243, \dots\}$.
- The common ratio is: 3.

- ◆ The function $f(n) = -(1 - 4n)$ represents a sequence. Create a table showing the first five terms in the sequence. Identify the domain and range of the function.

Solution:

n	1	2	3	4	5
$f(n)$	3	7	11	15	19

Since the function is a sequence, the domain would be n , the number of each term in the sequence. The set of numbers in the domain can be written as $\{1, 2, 3, 4, 5, \dots\}$. Notice that the domain is an infinite set of numbers, even though the table only displays the first five elements.

The range is $f(n)$ or (a_n) , the output numbers that result from applying the rule $-(1 - 4n)$. The set of numbers in the range, which is the sequence itself, can be written as $\{3, 7, 11, 15, 19, \dots\}$. This is also an infinite set of numbers, even though the table only displays the first five elements.

SAMPLE ITEMS

1. Look at the sequence in this table.

n	1	2	3	4	5	...
a_n	-1	1	3	5	7	...

Which function represents the sequence?

- A. $a_n = a_{n-1} + 1$
- B. $a_n = a_{n-1} + 2$
- C. $a_n = 2a_{n-1} - 1$
- D. $a_n = 2a_{n-1} - 3$

2. Which function is modeled in this table?

x	$f(x)$
1	8
2	11
3	14
4	17

- A. $f(x) = x + 7$
- B. $f(x) = x + 9$
- C. $f(x) = 2x + 5$
- D. $f(x) = 3x + 5$

3. Which explicit formula describes the pattern in this table?

d	C
2	6.28
3	9.42
5	15.70
10	31.40

- A. $d = 3.14 \times C$
- B. $3.14 \times C = d$
- C. $31.4 \times 10 = C$
- D. $C = 3.14 \times d$

4. If $f(12) = 4(12) - 20$, which function gives $f(x)$?

- A. $f(x) = 4x$
- B. $f(x) = 12x$
- C. $f(x) = 4x - 20$
- D. $f(x) = 12x - 20$

Answers to Unit 3.2 Sample Items

1. B 2. D 3. D 4. C

3.3 Interpret Functions That Arise in Applications in Terms of the Context

MGSE9-12.F.IF.4 Using tables, graphs, and verbal descriptions, interpret the key characteristics of a function which models the relationship between two quantities. Sketch a graph showing key features including: intercepts; intervals where the function is increasing, decreasing, positive, or negative; and end behavior.

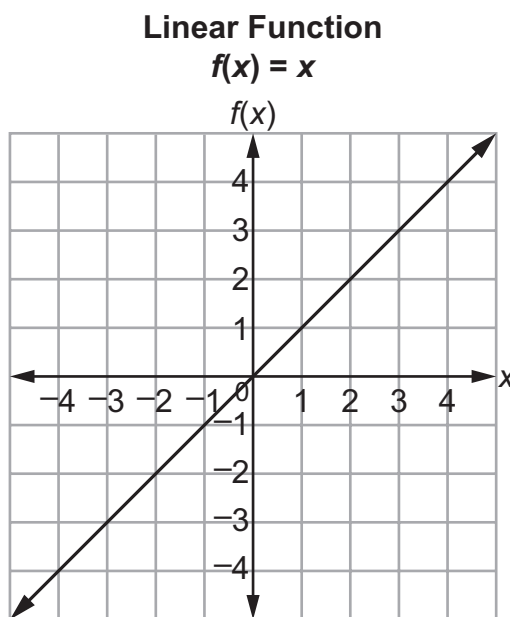
MGSE9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

MGSE9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

KEY IDEAS

By examining the graph of a function, many of its features are discovered. Features include domain and range, x - and y -intercepts, intervals where the function values are increasing or decreasing, positive or negative, and rates of change.

Example: Consider the graph of $f(x) = x$. It appears to be an unbroken line and slanted upward.

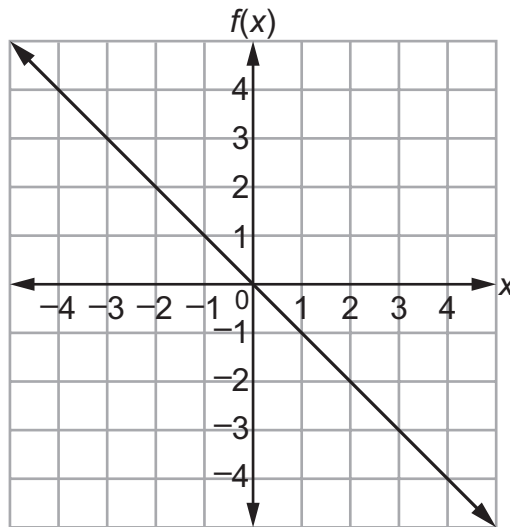


These are some of its key features:

- Domain: All real numbers
- Range: All real numbers
- x -intercept: The line appears to intersect the x -axis at 0
- y -intercept: The line appears to intersect the y -axis at 0
- Increasing: Always: as x increases, $f(x)$ increases
- Decreasing: Never
- Positive: $f(x)$ is positive when $x > 0$
- Negative: $f(x)$ is negative when $x < 0$

- Rate of change: 1
- End behavior: decreases as x goes to $-\infty$ and increases as x goes to ∞

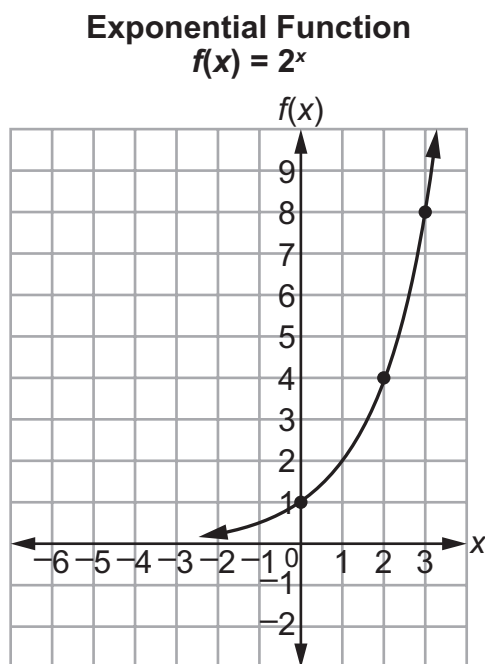
Example: Consider the graph of $f(x) = -x$. It appears to be an unbroken line and slanted downward.



These are some of its key features:

- Domain: All real numbers
- Range: All real numbers
- x -intercept: The line appears to intersect the x -axis at 0
- y -intercept: The line appears to intersect the y -axis at 0
- Increasing: Never
- Decreasing: Always: as x increases, $f(x)$ decreases
- Positive: $f(x)$ is positive when $x < 0$
Negative: $f(x)$ is negative when $x > 0$
- Rate of change: -1
- End behavior: increases as x goes to $-\infty$ and decreases as x goes to ∞

Example: Consider the graph of $f(x) = 2^x$.



These are some of its key features:

- Domain: All real numbers
- Range: $y > 0$
- x-intercept: None
- y-intercept: It appears to intersect the y-axis at 1
- Increasing: Always: as x increases, $f(x)$ increases
- Decreasing: Never
- Positive: $f(x)$ is positive for all x -values
- Negative: $f(x)$ is never negative
- Rate of change: Variable rate of change, as the graph represents a curve; the interval $1 \leq x \leq 2$ is approximately 2, but the rate of change is only 1 for $0 \leq x \leq 1$
- Asymptote: $y = 0$

Other features of functions can be discovered through examining their tables of values. The intercepts may appear in a table of values. From the differences of $f(x)$ -values over various intervals, we can tell if a function grows at a constant rate of change.

Example: Let $h(x)$ be the number of hours it takes a new factory to produce x engines. The company's accountant determines that the number of hours it takes depends on the time it takes to set up the machinery and the number of engines to be completed. It takes 6.5 hours to set up the machinery to make the engines and about 5.25 hours to completely manufacture one engine. The relationship is modeled with the function $h(x) = 6.5 + 5.25x$. Next, the accountant makes a table of values to check his function against his production records. The accountant starts with 0 engines because of the time it takes to set up the machinery.

The realistic domain for the accountant's function would be whole numbers, because you cannot manufacture a negative number of engines.

x, engines	$h(x)$, hours
0	6.5
1	11.75
2	17
3	22.25
4	27.5
5	32.75
10	59
100	531.5



From the table we can see the y -intercept. The y -intercept is the y -value when $x = 0$. The very first row of the table indicates the y -intercept is 6.5. Since we do not see a number 0 in the $h(x)$ column, we cannot tell from the table if there is an x -intercept. The x -intercept is the value when $h(x) = 0$.

$$\begin{aligned}h(x) &= 6.5 + 5.25x \\0 &= 6.5 + 5.25x \\-6.5 &= 5.25x \\-1.24 &= x\end{aligned}$$

The x -value when $y = 0$ is negative, which is not possible in the context of this example.

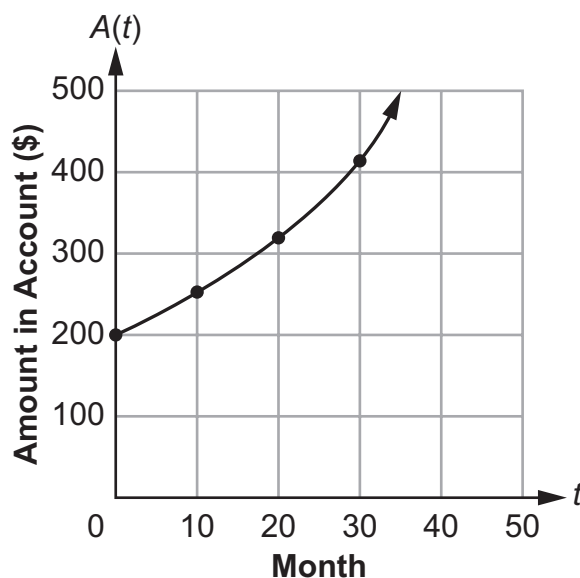
The accountant's table also gives us an idea of the rate of change of the function. We should notice that as x -values are increasing by 1, the $h(x)$ -values are growing by increments of 5.25. There appears to be a constant rate of change when the input values increase by the same amount. The increase from both 3 engines to 4 engines and 4 engines to 5 engines is 5.25 hours. The average rate of change can be calculated by comparing the values in the first or last rows of the table. The increase in number of engines manufactured is $100 - 0$, or 100. The increase in hours to produce the engines is $531.5 - 6.5$, or 525. The average rate of change is $\frac{525}{100} = 5.25$. The units for this average rate of change would be hours/engine, which happens to be the exact amount of time it takes to manufacture 1 engine.

Important Tips

-  You could begin exploration of a new function by generating a table of values using a variety of numbers from the domain. Decide, based on the context, what kinds of numbers can be in the domain, and make sure to choose negative numbers or numbers expressed as fractions or decimals if such numbers are included in the domain.
-  You cannot always find exact values from a graph. Always check your answers using the equation.

REVIEW EXAMPLES

- ◆ The amount accumulated in a bank account over a time period t and based on an initial deposit of \$200 is found using the formula $A(t) = 200(1.025)^t$, $t \geq 0$. Time, t , in months, is represented on the horizontal axis. The accumulated amount, $A(t)$, is represented on the vertical axis.



- What are the intercepts of the function?
- What is the domain of the function?
- Why are all the t -values nonnegative?
- What is the range of the function?

Solution:

There is no t -intercept because the account begins at 0 months when there is already \$200. Since t represents the time, in months, the account does not decrease below \$0. The function crosses the vertical axis at 200.

- There is no t -intercept because the bank account was never lower than \$200. The function crosses the vertical axis at 200.
- The domain is $t \geq 0$.
- The t -values are all nonnegative because they represent time, and time cannot be negative.
- $A(t) \geq 200$

- ◆ A company uses the function $V(x) = 28,000 - 1,750x$ to represent the amount left to pay on a truck, where $V(x)$ is the amount left to pay on the truck, in dollars, and x is the number of months after its purchase. Use the table of values shown below.

x (months)	$V(x)$ (\$)
0	28,000
1	26,250
2	24,500
3	22,750
4	21,000
5	19,250

- What is the y -intercept of the graph of the function in terms of the amount left to pay on the truck?
- Does the graph of the function have an x -intercept, and if so, what does it represent?
- Does the function increase or decrease?

Solution:

- From the table, when $x = 0$, $V(x) = 28,000$. So, the y -intercept is 28,000, which means at zero months, the amount left to pay on the truck had not yet decreased.
- Yes, it does have an x -intercept, although it is not shown in the table. The x -intercept is the value of x when $V(x) = 0$.

$$0 = 28,000 - 1,750x$$

$$-28,000 = -1,750x$$

$$16 = x$$

The x -intercept is 16. This means that the truck is fully paid off after 16 months of payments.

- For $x > 0$, as x increases, $V(x)$ decreases. Therefore, the function decreases.

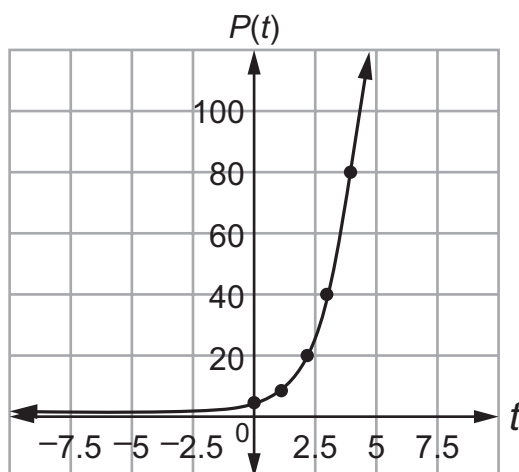
SAMPLE ITEMS

1. A wild horse runs at a rate of 8 miles an hour for 6 hours. Let y be the distance, in miles, the horse travels for a given amount of time, x , in hours. This situation can be modeled by a function.

Which of these describes the domain of the function?

- A. $0 \leq x \leq 6$
- B. $0 \leq y \leq 6$
- C. $0 \leq x \leq 48$
- D. $0 \leq y \leq 48$

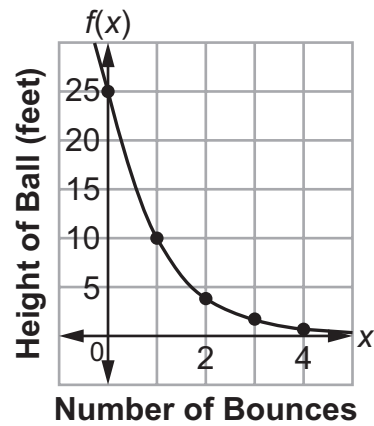
2. A population of squirrels doubles every year. Initially, there were 5 squirrels. A biologist studying the squirrels created a function to model their population growth, $P(t) = 5(2^t)$, where t is time. The graph of the function is shown.



Which values best describe the range of the population?

- A. any real number
- B. any whole number greater than 0
- C. any whole number greater than 5
- D. any whole number greater than or equal to 5

3. The function graphed on this coordinate grid shows $f(x)$, the height of a dropped ball, in feet, after its x th bounce.



On which bounce was the height of the ball 10 feet?

- A. bounce 1
- B. bounce 2
- C. bounce 3
- D. bounce 4

Answers to Unit 3.3 Sample Items

1. A 2. D 3. A

3.4 Analyze Functions Using Different Representations

MGSE9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology.

MGSE9-12.F.IF.7a Graph linear functions and show intercepts, maxima, and minima (as determined by the function or by context).

MGSE9-12.F.IF.7e Graph exponential functions, showing intercepts and end behavior.

MGSE9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one function and an algebraic expression for another, say which has the larger maximum.*

KEY IDEAS

When working with functions, it is essential to be able to interpret the specific quantitative relationship regardless of the manner of its presentation. Understanding different representations of functions, such as tables, graphs, equations, and verbal descriptions, makes interpreting relationships between quantities easier. Beginning with lines, we will learn how each representation aids our understanding of a function. Almost all lines are functions; vertical lines are an exception because they assign multiple elements of their range to just one element in their domain. All linear functions can be written in the form $y = mx + b$, where m and b are real numbers and x is a variable to which the function f assigns a corresponding value, $f(x)$.

Example: Consider the linear functions $f(x) = x + 5$, $g(x) = 2x - 5$, and $h(x) = -2x$.

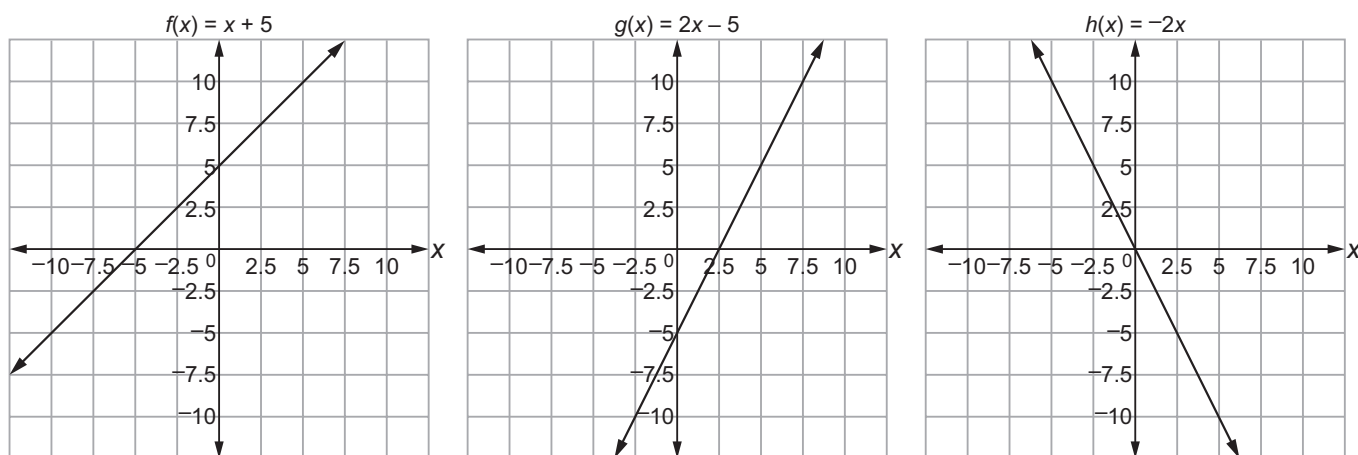
First, we will make a table of values for each equation. To begin, we need to decide on the domains. In theory, $f(x)$, $g(x)$, and $h(x)$ can accept any number as input. So the three of them have all real numbers as their domains. But for a table, we can only include a few elements of their domains. We should choose a sample that includes negative numbers, 0, and positive numbers. Place the elements of the domain in the left column, usually in ascending order. Then apply the function rule to determine the corresponding elements in the range. Place them in the right column.

x	$f(x) = x + 5$	x	$g(x) = 2x - 5$	x	$h(x) = -2x$
-3	2	-3	-11	-3	6
-2	3	-2	-9	-2	4
-1	4	-1	-7	-1	2
0	5	0	-5	0	0
1	6	1	-3	1	-2
2	7	2	-1	2	-4
3	8	3	1	3	-6
4	9	4	3	4	-8

We can note several features about the functions just from their tables of values.

- $f(x)$ has a y-intercept of 5. When x is 0, $f(x) = 5$. It is represented by (0, 5) on its graph.
- $g(x)$ has a y-intercept of -5 . When x is 0, $g(x) = -5$. It is represented by (0, -5) on its graph.
- $h(x)$ has a y-intercept of 0. When x is 0, $h(x) = 0$. It is represented by (0, 0) on its graph.
- $h(x)$ has an x-intercept of 0. When $h(x) = 0$, $x = 0$. It is represented by (0, 0) on its graph.
- $f(x)$ has an average rate of change of 1. $\frac{9 - 2}{4 - (-3)} = 1$
- $g(x)$ has an average rate of change of 2. $\frac{3 - (-11)}{4 - (-3)} = 2$
- $h(x)$ has an average rate of change of -2 . $\frac{(-8) - 6}{4 - (-3)} = -2$

Now we will take a look at the graphs of $f(x)$, $g(x)$, and $h(x)$.



The graphs confirm what we already knew about the functions' intercepts and their constant rates of change. To confirm, we can see that $f(x)$ increases by 2.5 as x increases by 2.5, which is a 1-to-1 rate of change. So the slope of $f(x)$ is 1. $g(x)$ increases by 10 as x increases by 5, which is a 2-to-1 rate of change. So the slope is 2. $h(x)$ decreases by 10 as x increases by 5, which is a -2 -to-1 rate of change. So the slope is -2 . The graphs suggest other information:

- $f(x)$ appears to have positive values for $x > -5$ and negative values for $x < -5$.
- $f(x)$ appears to be always increasing with no maximum or minimum values.
- $g(x)$ appears to have positive values for $x > 2.5$ and negative values for $x < 2.5$.
- $g(x)$ appears to be always increasing with no maximum or minimum values.
- $h(x)$ appears to have positive values for $x < 0$ and negative values for $x > 0$.
- $h(x)$ appears to be always decreasing with no maximum or minimum values.

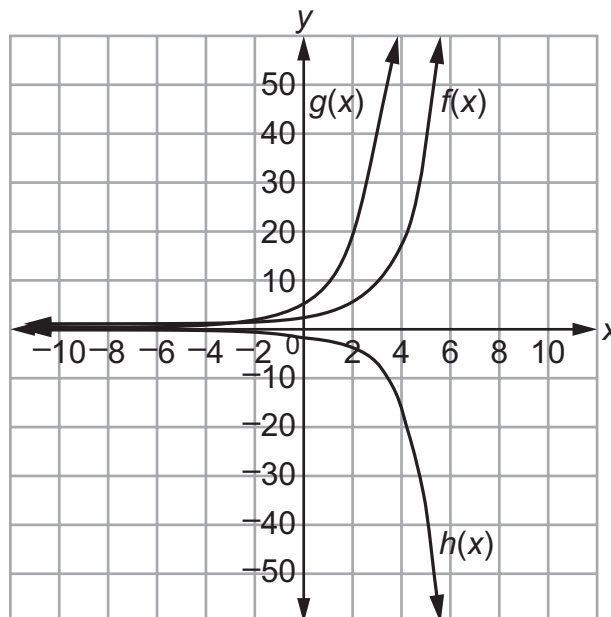
To confirm these observations, we can work with the equations for the functions. We suspect $f(x)$ is positive for $x > -5$. Since $f(x)$ is positive whenever $f(x) > 0$, write the inequality $x + 5 > 0$ and solve for x . We get $f(x) > 0$ when $x > -5$. We can confirm all our observations about $f(x)$ from working with the equation. Likewise, the observations about $g(x)$ and $h(x)$ can be confirmed using their equations.

Now let's represent $f(x) = 2x + 5$ contextually. Let $f(x)$ be the number of songs collected and x be the number of months the songs are collected. The information provided in terms of songs and months is represented differently while using the key features used with tables and graphs.

- There were initially 5 songs prior to starting the song collection.
- The collection of songs increases by 2 songs each month.
- The number of songs in the collection keeps increasing for as long as the songs are collected.
- There is no maximum value unless the songs are no longer collected.
- The minimum will not be lower than 5 songs.

The different ways of representing a function also apply to exponential functions. Exponential functions are built using powers. A power is the combination of a base with an exponent. For example, in the power 5^3 , the base is 5 and the exponent is 3. A function with a power where the exponent is a variable is an exponential function. Exponential functions are of the form $f(x) = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. In an exponential function, the base, b , is a constant and a is the coefficient.

Example: Consider $f(x) = 2^x$, $g(x) = 5 \cdot 2^x$, and $h(x) = -1 \cdot 2^x$. For all three functions, $f(x)$, $g(x)$, and $h(x)$, the base is 2. The coefficient in $f(x)$ is 1, $g(x)$ is 5, and $h(x)$ is -1 . The values of the coefficients cause the graphs to transform.



From the graphs, you can make the following observations:

	$f(x)$	$g(x)$	$h(x)$
y-intercept	$y = 1$	$y = 5$	$y = -1$
As x increases	$f(x)$ increases	$g(x)$ increases	$h(x)$ decreases
As x decreases	$f(x)$ approaches 0	$g(x)$ approaches 0	$h(x)$ approaches 0
Domain	all real numbers	all real numbers	all real numbers
Range	all real numbers greater than 0	all real numbers greater than 0	all real numbers less than 0
Note: None of the functions have a constant rate of change. All of the functions have an asymptote of $y = 0$.			

Now look at tables of values for these functions.

x	$f(x) = 2^x$	x	$g(x) = 5 \cdot 2^x$	x	$h(x) = -1 \cdot 2^x$
-3	$\frac{1}{8}$	-3	$\frac{5}{8}$	-3	$-\frac{1}{8}$
-2	$\frac{1}{4}$	-2	$\frac{5}{4}$	-2	$-\frac{1}{4}$
-1	$\frac{1}{2}$	-1	$\frac{5}{2}$	-1	$-\frac{1}{2}$
0	1	0	5	0	-1
1	2	1	10	1	-2
2	4	2	20	2	-4
3	8	3	40	3	-8
4	16	4	80	4	-16

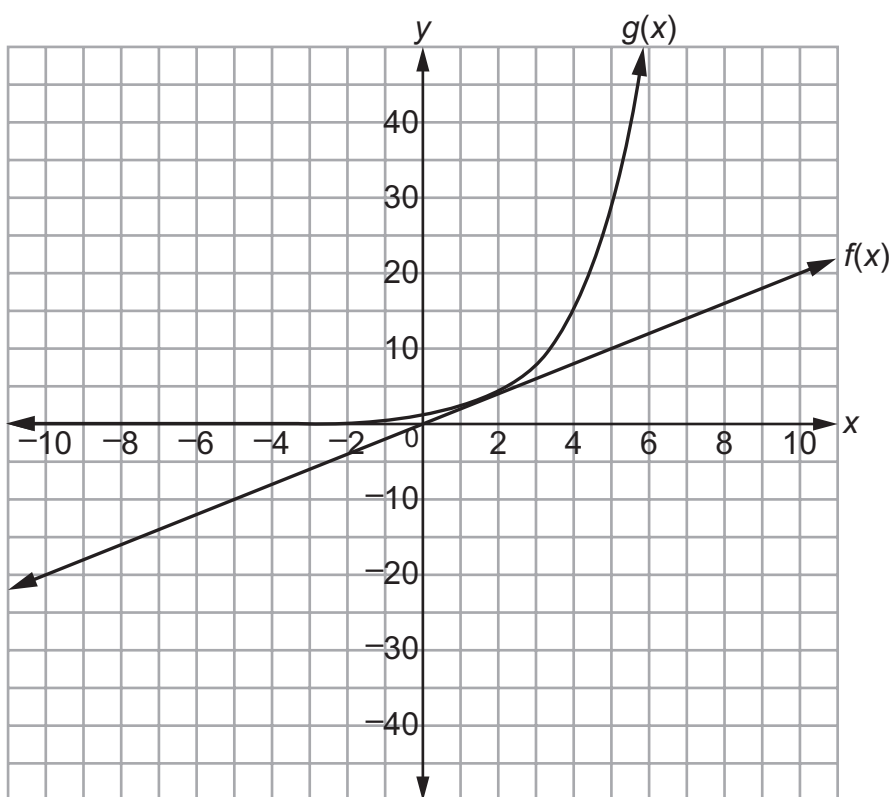
The tables confirm all three functions have y -intercepts: $f(0) = 1$, $g(0) = 5$, and $h(0) = -1$. Although the tables do not show a constant rate of change for any of the functions, a rate of change can be determined on a specific interval by finding the change in the y -value divided by the change in the x -value for two distinct points on a graph.

Now let's represent $g(x) = 5 \cdot 2^x$ contextually. Let $g(x)$ be the population of bacteria and x be the number of days the bacteria population increases. The information provided in terms of bacteria and days is represented differently while using the key features used with tables and graphs.

- There were initially 5 bacteria prior to the population of bacteria increasing.
- The bacteria double each day.
- The bacteria population increases as the number of days increases.
- There is no maximum value.
- The minimum value is 5 bacteria.
- The number of bacteria will always range from 5 to infinity.

Comparing functions helps us gain a better understanding of them. Let's take a look at a linear function and the graph of an exponential function.

Consider $f(x) = 2x$ and the graph of $g(x)$ shown. The function $f(x) = 2x$ represents a linear relationship. The graph of $g(x)$ shows an exponential relationship. We know the linear function has a graph that is a straight line and has a constant rate of change. The graph of the exponential function is curved and has a varying rate of change. Both graphs will have y -intercepts. The exponential curve appears to have a y -intercept below 5. The y -intercept of the line is $f(0) = 2 \cdot 0 = 0$. Since $f(0) = 0$, the line must pass through the point $(0, 0)$, so 0 must also be the x -intercept of the line. The graph of the exponential function does not appear to have an x -intercept, though the curve appears to come very close to the x -axis.



For $f(x) = 2x$, the domain and range are all real numbers. For the exponential function, the domain is defined for all real numbers and the range is defined for positive values since the asymptote, where $g(x)$ approaches as the value of x decreases, is at $y = 0$.

So while the two graphs share some features, they also have significant differences, the most important of which is that the linear function is a straight line and has a constant rate of change, while the exponential function is curved with a rate of change that is increasing.

Important Tips

- Remember that the elements of the domain and the values obtained by substituting them into the function rule form the coordinates of the points that lie on the graph of a function. The domain and range can also be determined by examining the graph of a function by looking for asymptotes on the graph of an exponential function or looking for endpoints or continuity for linear and exponential functions.
- Be familiar with important features of a function such as intercepts, domain, range, minima and maxima, end behavior, asymptotes, and periods of increasing and decreasing values.

REVIEW EXAMPLES

- ◆ What are the key features of the function $p(x) = \frac{1}{2}x - 3$?

Solution:

First, notice that the function is linear. The domain for the function is the possible numbers we can substitute for x . Since the function is linear, and is not related to a real-life situation where certain values are not applicable, the domain is all real numbers. The graphic representation will give us a better idea of its range.

We can determine the y -intercept by finding $p(0)$:

$$p(0) = \frac{1}{2}(0) - 3 = -3$$

So, the graph of $p(x)$ will intersect the y -axis at $(0, -3)$. To find the x -intercept, we have to solve the equation $p(x) = 0$.

$$\frac{1}{2}x - 3 = 0$$

$$\frac{1}{2}x = 3$$

$$x = 6$$

So, the x -intercept is 6. The line intersects the x -axis at $(6, 0)$.

Now we will make a table of values to investigate the rate of change of $f(x)$.

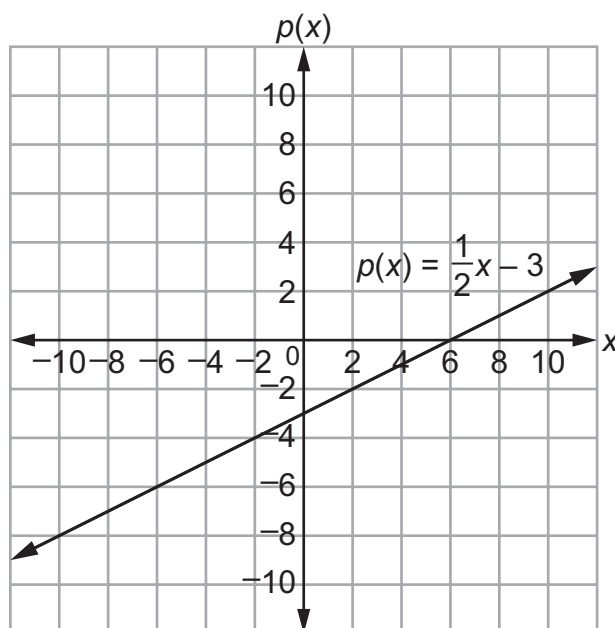
x	$p(x) = \frac{1}{2}x - 3$
-3	$-\frac{9}{2}$
-2	-4
-1	$-\frac{7}{2}$
0	-3
1	$-\frac{5}{2}$
2	-2
3	$-\frac{3}{2}$
4	-1

Notice the row that contains the values 0 and -3 . These numbers correspond to the point where the line intersects the y -axis, confirming that the y -intercept is -3 . Since 0 does not appear in the right column, the coordinates of the x -intercept point are not in the table of values. We notice that the values in the right column keep increasing by $\frac{1}{2}$. We can calculate the average rate of change.

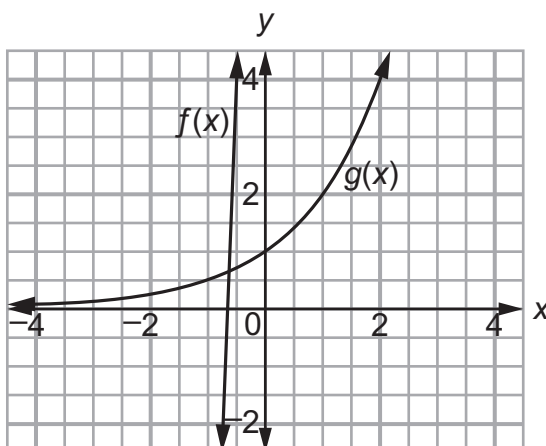
$$\text{Average rate of change: } \frac{-1 - \frac{-9}{2}}{4 - (-3)} = \frac{1}{2}$$

It turns out the average rate of change is the same as the incremental differences in the outputs. This confirms the function $p(x)$ has a constant rate of change. Notice that $\frac{1}{2}$ is the coefficient of x in the function rule.

Now we will examine the graph. The graph shows a line that appears to be always increasing. Since the line has no minimum or maximum value, its range would be all real numbers. The function appears to have positive values for $x > 6$ and negative values for $x < 6$.



- ◆ Consider the graphs of $f(x)$ and $g(x)$. Both functions represent money increasing in two different accounts.



Although we are not able to see the y -intercept of f in the graph, we can determine the y -intercept if given an equation. In this model, the y -intercept can be seen for g but we are not able to determine the growth factor.

In this situation, one account increases at a rate of \$30 per week. The account at week zero contained \$20. So, the function $f(x) = 30x + 20$ after x weeks represents this situation.

The second account doubles each week. The account at week zero only contained \$1. So, the function $g(x) = 2^x$ after x weeks represents this situation.

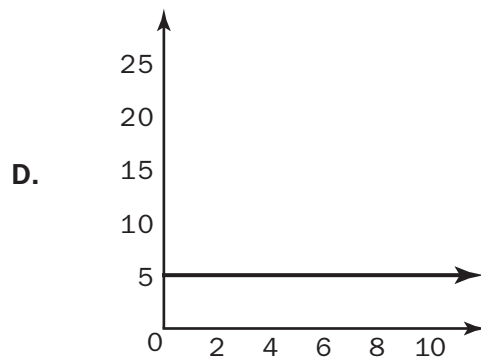
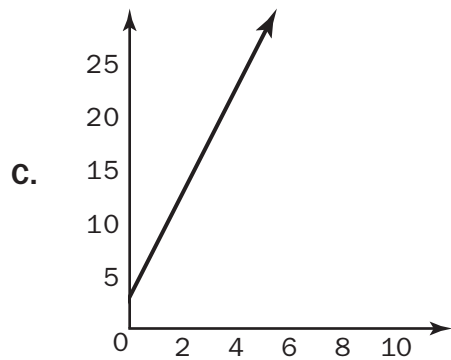
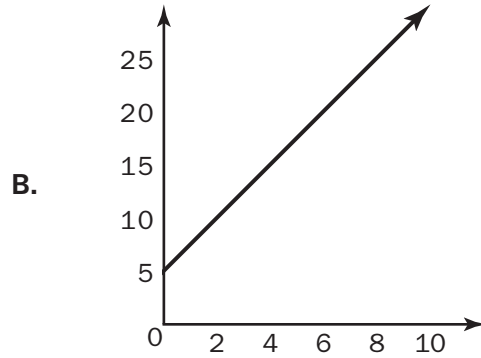
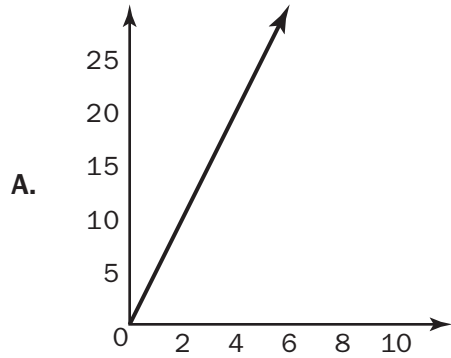
Since either account did not have a negative balance, we can set the domain of the function for both accounts to be all real numbers ≥ 0 . From the graph or the functions, which account initially contains more money? We can find $f(0)$ and $g(0)$ to get $f(0) = 30(0) + 20 = 20$ and $g(0) = 2^0 = 1$. So, the first account initially has more money. This is also the y -intercept.

Both graphs appear to be increasing as x increases. This means the money in both accounts increases as the number of weeks increase. So we can say the range for the first account is all real numbers ≥ 20 and the range for the second account is all real numbers ≥ 1 .

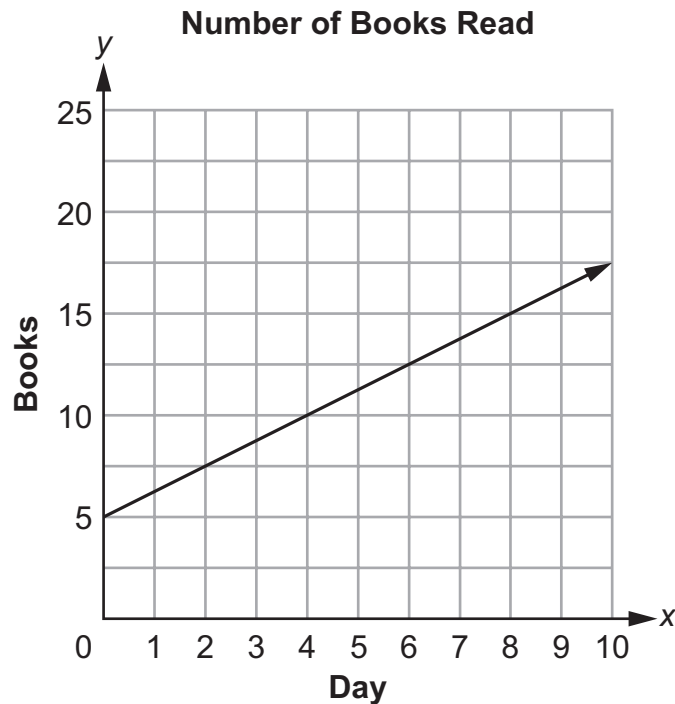
At what rate is each account increasing? The first account increases at an average rate of \$30 per week. The second account increases by a factor of 2 each week.

SAMPLE ITEMS

1. To rent a canoe, the cost is \$3 for the oars and life preserver, plus \$5 an hour for the canoe. Which graph models the cost of renting a canoe?



2. Juan and Patti decided to see who could read the most books in a month. They began to keep track after Patti had already read 5 books that month. This graph shows the number of books Patti read for the next 10 days.



If Juan has read no books before the fourth day of the month and he reads at the same rate as Patti, how many books will he have read by day 12?

- A. 5
- B. 10
- C. 15
- D. 20

Answers to Unit 3.4 Sample Items

1. C 2. B

3.5 Build a Function That Models a Relationship between Two Quantities

MGSE9-12.F.BF.1 Write a function that describes a relationship between two quantities.

MGSE9-12.F.BF.1a Determine an explicit expression and the recursive process (steps for calculation) from context. *For example, if Jimmy starts out with \$15 and earns \$2 a day, the explicit expression “ $2x+15$ ” can be described recursively (either in writing or verbally) as “to find out how much money Jimmy will have tomorrow, you add \$2 to his total today.”*

MGSE9-12.F.BF.2 Write arithmetic and geometric sequences recursively and explicitly, use them to model situations, and translate between the two forms. Connect arithmetic sequences to linear functions and geometric sequences to exponential functions.

KEY IDEAS

Modeling a quantitative relationship can be a challenge. But there are some techniques we can use to make modeling easier. Functions can be written to represent the relationship between two variables.

Example: Joe started with \$13 saved. He has been saving \$2 each week to purchase a baseball glove. The amount of money Joe has saved depends on how many weeks he has been saving. This means the money he has saved is the dependent variable and the number of weeks is the independent variable. So the number of weeks and the amount Joe has saved are related. We can begin with the function $S(x)$, where $S(x)$ is the amount he has saved and x is the number of weeks. Since we know that he started with \$13 saved and that he saves \$2 each week, we can use a linear model, one where the change is constant.

A linear model for a function is $y = mx + b$, where a and b are any real numbers and x is the independent variable.

So the model is $S(x) = 2x + 13$, which will generate the amount Joe has saved after x weeks.

Exponential decay is when a value decreases by a common factor. For a function $f(x) = ab^x$, the **decay factor** is b when $a > 0$ and $0 < b < 1$. This means that a decreases by b times as x increases. The decay rate is the rate at which $f(x)$ decreases. In the next example, you will see that the decay factor is $\frac{1}{2}$.

Exponential functions can be used to model quantitative relationships. These functions can be written to represent a relationship between two variables and are sometimes referred to as a geometric sequence.

Example: Pete withdraws half his savings every week. If he started with \$400, can a rule be written for how much Pete has left each week? We know the amount Pete has left depends on the week. We can start with the amount Pete has, $A(x)$. The amount depends on the week number, x . However, the rate of change is not constant. Therefore, the previous method for finding a function will not work. We could set up the model as

$$A(x) = 400 \cdot \frac{1}{2} \cdot \dots \cdot \frac{1}{2}, \text{ with } \frac{1}{2} \text{ being multiplied as many times as the number of weeks, } x.$$

Or we can use a power of $\frac{1}{2}$:

$$A(x) = 400 \cdot \left(\frac{1}{2}\right)^x$$

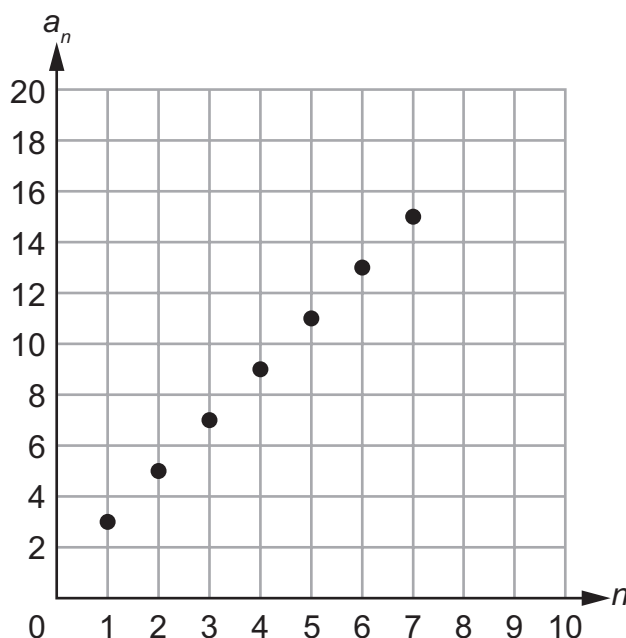
Note that the function assumes Pete had \$400 at week 0 and withdrew half during week 1. The exponential function will generate the amount Pete has after x weeks.

n	1	2	3	4	5	6	7
a_n	200	100	50	25	12.5	6.25	3.125
$\frac{a_n}{a_{n-1}}$	—	$\frac{100}{200} = \frac{1}{2}$	$\frac{50}{100} = \frac{1}{2}$	$\frac{25}{50} = \frac{1}{2}$	$12.5 \div 25 = \frac{1}{2}$	$6.25 \div 12.5 = \frac{1}{2}$	$3.125 \div 6.25 = \frac{1}{2}$

Sometimes the data for a function is presented as a sequence.

Example: Suppose we know the total number of cookies eaten by Rachel on a day-to-day basis over the course of a week. We might get a sequence like this: 3, 5, 7, 9, 11, 13, 15. There are two ways we could model this sequence. The first would be the explicit way. We would arrange the sequence in a table. Note that d in the third row means common difference.

n	1	2	3	4	5	6	7
a_n	3	5	7	9	11	13	15
d	—	$5 - 3 = 2$	$7 - 5 = 2$	$9 - 7 = 2$	$11 - 9 = 2$	$13 - 11 = 2$	$15 - 13 = 2$



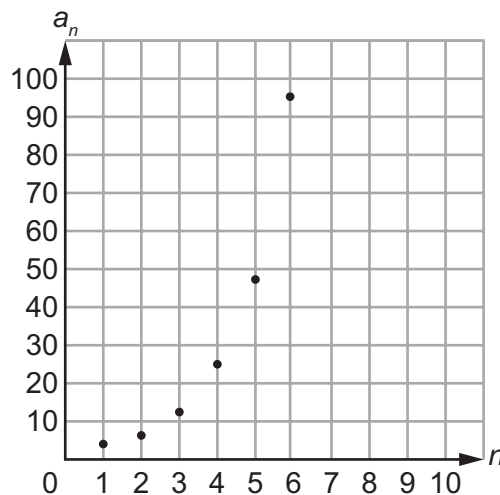
Since the difference between successive terms of the sequence is constant, namely 2, we can again use a linear model. But this time we do not know the y -intercept because there is no zero term ($n = 0$). However, if we work backward, a_0 —the term before the first—would be 1, so the starting number would be 1. That leaves us with an explicit formula: $f(n) = 2n + 1$, for $n > 0$ (n is an integer). The explicit formula $a_n = a_1 + d(n - 1)$ where a_1 is the first term and d is the common difference can be used to find the explicit function. A sequence that can be modeled with a linear function is called an **arithmetic sequence**.

Another way to look at the sequence is recursively. We need to express term a_n in terms of a previous term a_{n-1} . Since n is the term, then $n - 1$ is used to represent the previous term. For example, a_3 is the third term so $a_{3-1} = a_2$ is the second term. Since the constant difference is 2, we know $a_n = a_{n-1} + 2$ for $n > 1$, with $a_1 = 3$.

Some sequences can be modeled exponentially. For a sequence to fit an exponential model, the ratio of successive terms is constant. In the example below, notice the third row shows a constant ratio between consecutive terms.

Example: Consider the number of sit-ups Clara does each week as listed in the sequence 3, 6, 12, 24, 48, 96, 192. Clara is doing twice as many sit-ups each successive week. It might be easier to put the sequence in a table to analyze it.

n	1	2	3	4	5	6	7
a_n	3	6	12	24	48	96	192
$\frac{a_n}{a_{n-1}}$	—	$\frac{6}{3} = 2$	$\frac{12}{6} = 2$	$\frac{24}{12} = 2$	$\frac{48}{24} = 2$	$\frac{96}{48} = 2$	$\frac{192}{96} = 2$



The third row in the table shows how the common ratio is determined. A fraction $\frac{a_n}{a_{n-1}}$ is written. For example, $\frac{a_3}{a_2} = \frac{12}{6} = 2$. So the common ratio is 2. It appears as if each term is twice the term before it. But the difference between the terms is not constant. This type of sequence shows exponential growth. The function type is $f(x) = a(b^x)$. In this type of function, a is the coefficient, b^x is the growth factor, and b is the common ratio. The growth rate for this sequence is 100% since each term is doubled to get the next term. For the sequence above, the growth power is 2^x because the terms keep doubling. To find b , you need to know the first term. The first term is 3. The second term is the first term of the sequence multiplied by the common ratio once. The third term is the first term multiplied by the common ratio twice. Since that pattern continues, our exponential function is $f(x) = 3(2^{x-1})$. The function $f(x) = 3(2^{x-1})$ would be the **explicit** or **closed form** for the sequence. The explicit formula $a_n = a_1(r^{n-1})$, where a_1 is the first term and r

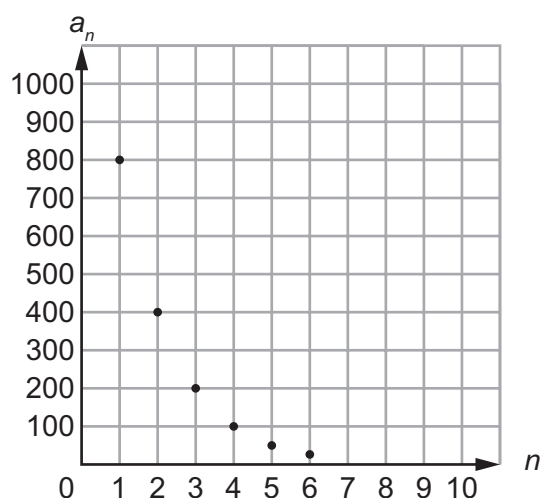
is the common ratio, can be used to determine the exponential equation. You can find this formula on the formula sheet. A sequence that can be modeled by an exponential function is a **geometric sequence**.

The sequence could also have a recursive rule. Since the next term is twice the previous term, the recursive rule would be $a_n = 2 \cdot a_{n-1}$, with a first term, a_1 , of 3.

Exponential functions have lots of practical uses. They are used in many real-life situations.



Example: A scientist collects data on the number of microbes in a colony over a number of days. She notes these numbers:

Day	1	2	3	4	5	6
a_n	800	400	200	100	50	25
$\frac{a_n}{a_{n-1}}$	—	$\frac{400}{800} = 0.5$	$\frac{200}{400} = 0.5$	$\frac{100}{200} = 0.5$	$\frac{50}{100} = 0.5$	$\frac{25}{50} = 0.5$



Notice a key feature can be seen from the graph. This graph represents exponential decay. Since the ratio between successive terms is 0.5, which is less than 1, the decay factor or common ratio is 0.5. From the table, we can see the initial term, $a_1 = 800$. Using the formula $a_n = a_1(r)^{(n-1)}$, we can determine the equation $a_n = 800(0.5)^{(n-1)}$. Since the decay factor is 0.5, the decay rate is 50%. This means the colony of microbes has a half-life of 1 day because it takes 1 day for the number of microbes to decrease by half.

Important Tips

-  Examine function values to draw conclusions about the rate of change.
-  Keep in mind the general forms of a linear function and exponential function.

REVIEW EXAMPLES

- ◆ The terms of a sequence increase by a constant amount. If the first term is 7 and the fourth term is 16:

- List the first six terms of the sequence.
- What is the explicit formula for the sequence?
- What is the recursive rule for the sequence?

Solution:

- The sequence would be 7, 10, 13, 16, 19, 22, . . . $\frac{16 - 7}{4 - 1} = \frac{9}{3} = 3$. If the difference between the first and fourth terms is 9, the constant difference is 3. So, the sequence is arithmetic.
- Since the constant difference is 3, $a = 3$. Because the first term is 7, $b = 7 - 3 = 4$. So, the explicit formula is $f(n) = 3(n) + 4$ for $n > 0$ or $a_n = a_{n-1} + 3$ for $n > 0$.
- Since the difference between successive terms is 3, $a_n = a_{n-1} + 3$ with $a_1 = 7$.

- ◆ Each week, Tim wants to increase the number of sit-ups he does daily by 2 sit-ups. The first week, he does 15 sit-ups each day.

Write an explicit function in the form $f(n) = mn + b$ to represent the number of sit-ups, $f(n)$, Tim does daily in week n .

Solution:

The difference between the number of daily sit-ups each week is always 2, so this is a linear model with $m = 2$. Since there is no zero term, we take the first term, ($n = 1$), and work backwards by subtracting 2 from 15. This gives us $b = 13$. Therefore, the explicit function is $f(n) = 2n + 13$.

A recursive function in the form $f(n) = f(n - 1) + d$, where $f(0) = a$, can be written for the sit-up problem. What recursive function represents the number of sit-ups, $f(n)$, Tim does daily in week n ?

Solution:

Since Tim starts out doing 15 sit-ups each day, $f(0) = 15$. The variable d stands for the difference between the number of daily sit-ups Tim does each week, which is 2. The recursive function will be $f(n) = f(n - 1) + 2$, where $f(0) = 15$.

- ◆ The temperature of a large tub of water that is currently at 100° F decreases by about 10% each hour.

Part A: Write an explicit function in the form $f(n) = a \cdot b^n$ to represent the temperature, $f(n)$, of the tub of water in n hours.

Solution:

The ratio between the changes in temperature each hour is 0.90 since the temperature is decreasing by 0.10. So, this is an exponential model with $0.90 = b$. Since the first term is 100, $a = 100$. Substitute the values into the function $f(n) = a \cdot b^{(n-1)}$, which gives the equation $f(n) = 100(0.90)^{(n-1)}$.

Part B: A recursive function in the form $f(n) = r(f(n - 1))$, where $f(1) = 100$, can be written for the temperature problem. What recursive function represents the temperature, $f(n)$, of the tub in hour n ?

Solution:

The temperature starts at $f(1) = 100$. The variable r stands for the ratio between subsequent temperatures, which is 0.90. The recursive function will be $f(n) = (0.90)f(n - 1)$ for all $n > 1$.

SAMPLE ITEMS

1. Which function represents this sequence?

n	1	2	3	4	5	...
a_n	6	18	54	162	486	...

- A. $f(n) = 3^{n-1}$
- B. $f(n) = 6^{n-1}$
- C. $f(n) = 3(6^{n-1})$
- D. $f(n) = 6(3^{n-1})$

2. The first term in this sequence is 3.

n	1	2	3	4	5	...
a_n	3	10	17	24	31	...

Which function represents the sequence?

- A. $f(n) = n + 3$
- B. $f(n) = 7n - 4$
- C. $f(n) = 3n + 7$
- D. $f(n) = n + 7$

3. The points (0, 1), (1, 5), (2, 25), and (3, 125) are on the graph of a function. Which equation represents that function?

- A. $f(x) = 2^x$
- B. $f(x) = 3^x$
- C. $f(x) = 4^x$
- D. $f(x) = 5^x$

Answers to Unit 3.5 Sample Items

1. D 2. B 3. D

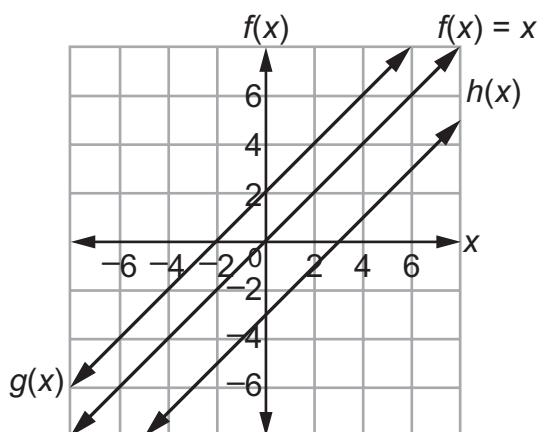
3.6 Build New Functions from Existing Functions

MGSE9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *(Focus on vertical translations of graphs of linear and exponential functions. Relate the vertical translation of a linear function to its y-intercept.)*

KEY IDEAS

Functions can be transformed in many ways. Whenever a function rule is transformed, the transformation affects the graph. One way to change the effect on the characteristics of a graph is to change the constant value of the function. Such adjustments have the effect of shifting the function's graph up or down or right or left. These shifts are called **translations**. The original graph is moved from one place to another on the coordinate plane the same as translations in geometry.

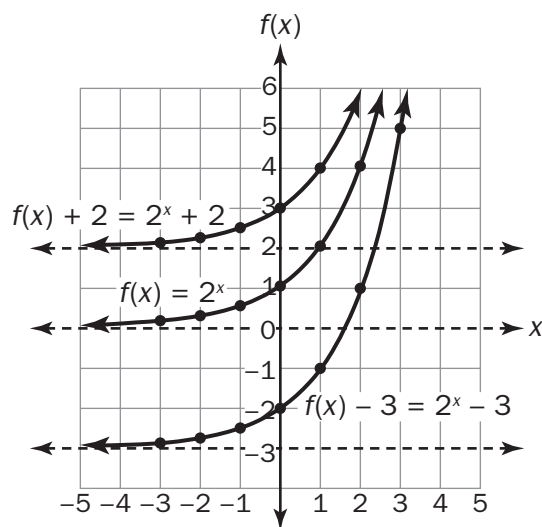
Example: If $f(x) = x$, how will $g(x) = f(x) + 2$ and $h(x) = f(x) - 3$ compare?



As expected, the lines all have the same shape, rate of change, domain, and range. The adjustments affected the intercepts. When 2 is added to the function's values, the y-intercept is 2 units higher. When 3 is subtracted, the y-intercept drops 3 units. When quantities are added or subtracted from a function's values, it causes a vertical translation of the graph. This effect would be true of all types of functions.

Example: If $f(x) = 2^x$, how will $g(x) = f(x) + 2$ and $h(x) = f(x) - 3$ compare?

By using substitution, it makes sense that if $f(x) = 2^x$, then $g(x) = f(x) + b = 2^x + b$. We will compare graphs of $f(x) = 2^x$, $g(x) = f(x) + 2 = 2^x + 2$, and $h(x) = f(x) - 3 = 2^x - 3$.



The curves have not changed shape. Their domains are unchanged. However, the curves are shifted vertically. The function $g(x)$ is a translation of $f(x) = 2^x$ upward by 2 units. The function $h(x)$ is a translation downward by 3 units. The asymptotes are also shifted vertically.

REVIEW EXAMPLES

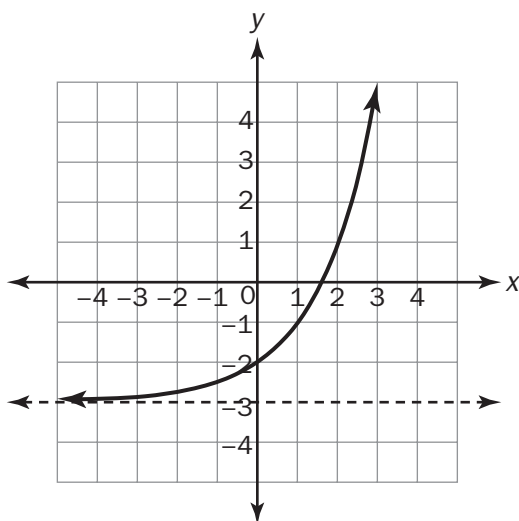
◆ For the function $f(x) = 3^x$:

Find the function that represents a 5-unit upward translation of the function.

Solution:

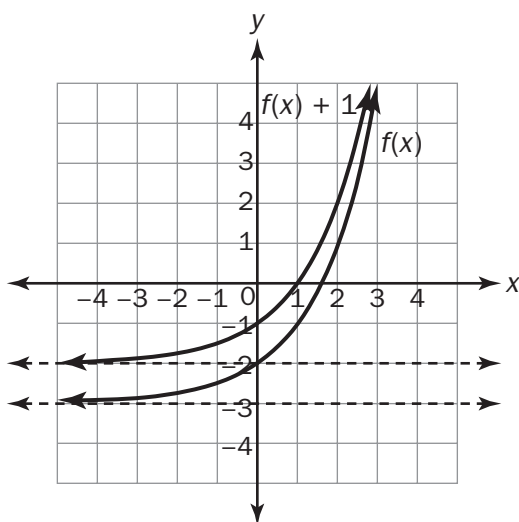
$$f(x) = 3^x + 5$$

- ◆ The graph of function $f(x) = 2^x - 3$ is shown below.



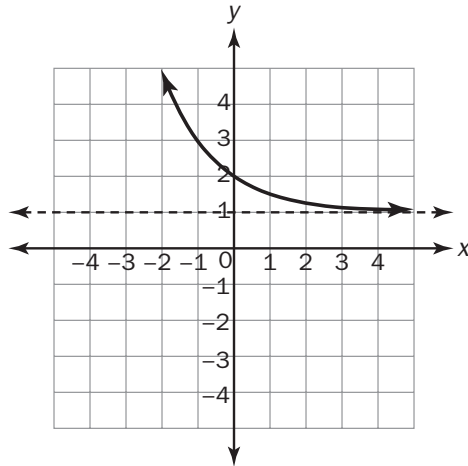
Draw the graph of $f(x) + 1$. Draw a dashed line to indicate the minimum or maximum of the function.

Solution:



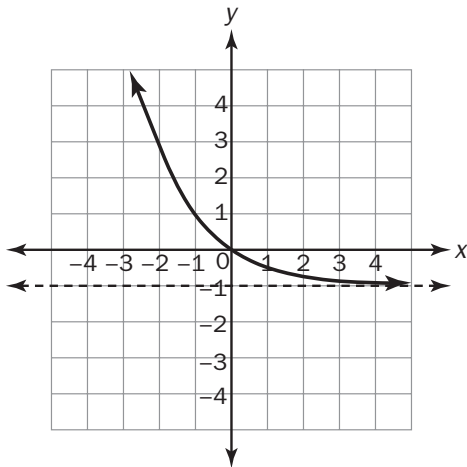
SAMPLE ITEMS

1. The function $f(x)$ is graphed below.

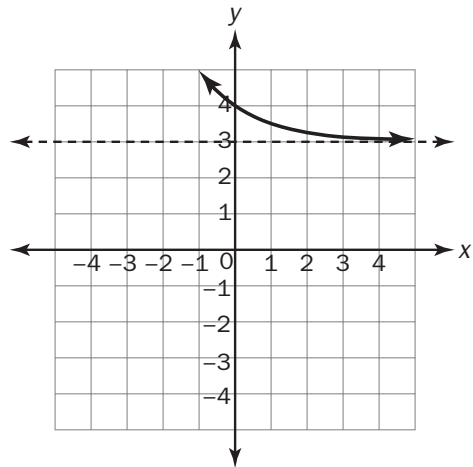


Which graph shows $f(x) + 2$?

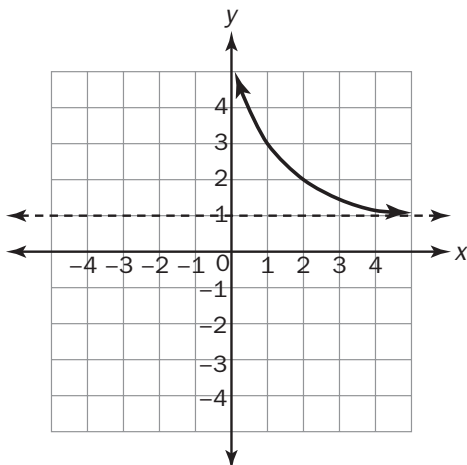
A.



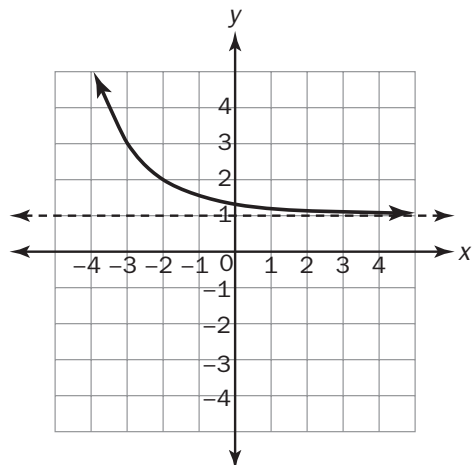
B.



C.



D.



2. Which graph shows the function $f(x) = 2x$ being translated 4 units down?

- A. graph of linear function $g(x) = 2x + 8$
- B. graph of linear function $g(x) = 2x - 8$
- C. graph of linear function $g(x) = 2x + 4$
- D. graph of linear function $g(x) = 2x - 4$

Answers to Unit 3.6 Sample Items

1. B 2. D

3.7 Construct and Compare Linear and Exponential Models and Solve Problems

MGSE9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

MGSE9-12.F.LE.1a Show that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals. (This can be shown by algebraic proof, with a table showing differences, or by calculating average rates of change over equal intervals.)

MGSE9-12.F.LE.1b Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

MGSE9-12.F.LE.1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

MGSE9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

MGSE9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly.

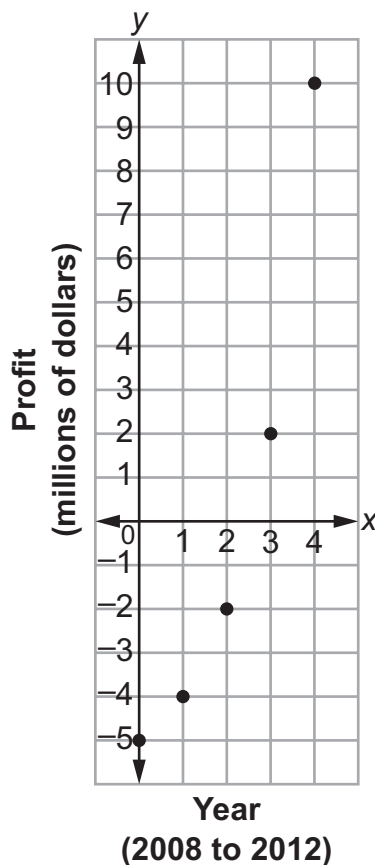
KEY IDEAS

Recognizing linear growth and exponential growth rates is key to modeling a quantitative relationship. The most common growth in nature is either linear or exponential. Linear growth happens when the dependent variable changes are the same for equal intervals of the independent variable. Exponential growth happens when the dependent variable changes at the same percent rate for equal intervals of the independent variable.

Example: Given a table of values, look for a constant rate of change in the y , or $f(x)$, column. The table below shows a constant rate of change, namely -2 , in the $f(x)$ column for each unit change in the independent variable x . The table also shows the y -intercept of the relation. The function has a y -intercept of $+1$, the $f(0)$ -value. These two pieces of information allow us to find a model for the relationship. When the change in $f(x)$ is constant, we use a linear model, $y = mx + b$, where a represents the constant rate of change and b the y -intercept. For the given table, the m -value is -2 , the constant change in the $f(x)$ -values, and b is the y -intercept, the value of the function at $x = 0$. The function is $f(x) = -2x + 1$. Using the linear model, we are looking for an explicit formula for the function.

x	$f(x)$	Change in $f(x)$
-2	5	—
-1	3	$3 - 5 = -2$
0	1	$1 - 3 = -2$
1	-1	$-1 - 1 = -2$
2	-3	$-3 - (-1) = -2$

Example: Given the graph below, compare the coordinates of the points to determine whether there is linear or exponential growth.



The points represent the profit/loss of a new company over its first 5 years, from 2008 to 2012. The company started out \$5,000,000 in debt. After 5 years, it had a profit of \$10,000,000. From the arrangement of the points, the pattern does not look linear. We can check by considering the coordinates of the points and using a table of values.

x	y	Change in y
0	-5,000,000	—
1	-4,000,000	$-4,000,000 - (-5,000,000) = 1,000,000$
2	-2,000,000	$-2,000,000 - (-4,000,000) = 2,000,000$
3	2,000,000	$2,000,000 - (-2,000,000) = 4,000,000$
4	10,000,000	$10,000,000 - (2,000,000) = 8,000,000$

The y changes are not constant for equal x intervals. Therefore, it is confirmed that this is not linear. However, the ratios of successive differences are equal.

$$\frac{2,000,000}{1,000,000} = \frac{4,000,000}{2,000,000} = \frac{2}{1} = 2$$

Having a constant percent for the growth rate for equal intervals indicates exponential growth. The relationship can be modeled using an exponential function. Since the common factor is 2, we can begin with the function $f(x) = 2^x$. However, our example does not cross the y-axis at 1, or 1,000,000 as does $f(x) = 2^x$. Since the initial profit value was not \$1,000,000, the exponential function has been translated downward. The amount of the translation is \$6,000,000. We model the company's growth as:

$$P(x) = 1,000,000(2^x) - 6,000,000$$

The $-6,000,000$ helps us find the horizontal asymptote at $y = -6,000,000$.





We can use our analysis tools to compare growth. For example, it might be interesting to consider whether you would like your pay raises to be linear or exponential. Linear growth is characterized by a constant number. With linear growth, a value grows by the same amount each time. Exponential growth is characterized by a percent which is called the growth rate.

Example: Suppose you start work and earn \$30,000 per year. After one year, you are given two choices for getting a raise: a) a 2% per year raise or b) 600 dollars for the first year plus a flat \$15 per year raise for each successive year. Which option is better? We can make a table with both options and see what happens.

Year	Yearly Pay	
	2% per year raise	\$15 per year raise
0	\$600	\$600
1	\$612	\$615
2	\$624.24	\$630
3	\$636.72	\$645

Looking at years 1 through 3, the \$15 per year option seems better. However, look closely at the 2% column. Though the pay increases start out smaller each year, they are growing exponentially. For some year in the future, the 2% per year increase in salary will be more than the \$15 per year increase in salary. If you know the number of years you expect to work at the company, it will help determine which option is best.

Important Tips

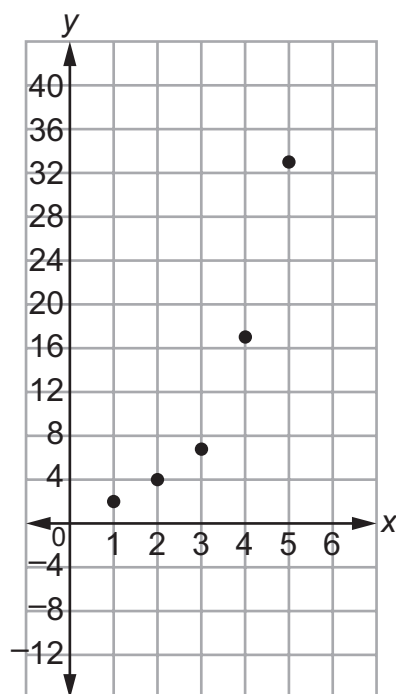
-  Examine function values carefully.
-  Remember that a linear function has a constant rate of change.
-  Keep in mind that growth rates are modeled with exponential functions.
-  Growth rate is not the same as growth factor. In the example, the growth rate of 2% does not equal a growth factor of 2. The growth factor in this case is 1.02.

REVIEW EXAMPLES

- ◆ The swans on Elsworth Pond have been increasing in number each year. Felix has been keeping track, and so far he has counted 2, 4, 7, 17, and 33 swans each year for the past 5 years.
- Make a scatter plot of the swan population.
 - What type of model would be a better fit, linear or exponential? Explain your answer.
 - How many swans should Felix expect next year if the trend continues? Explain your answer.

Solution:

a.



- Exponential; the swan population does not grow by the same number each year. The swan population appears to be nearly doubling every year.
- There could be about 64 swans next year. A function modeling the swan growth would be $P(x) = 2^x$, which would predict $P(6) = 2^6 = 64$.

- ◆ Given the sequence 7, 10, 13, 16, . . .

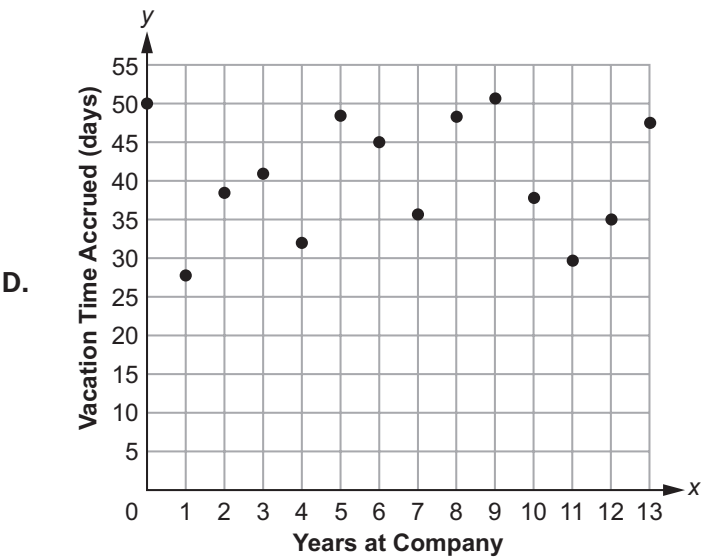
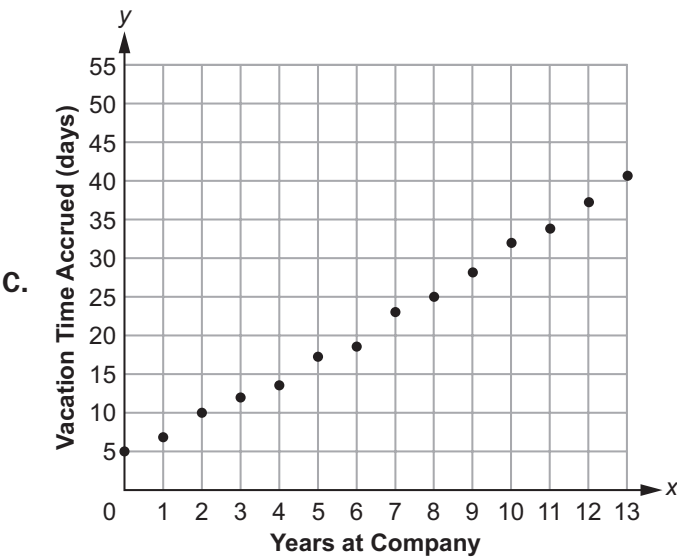
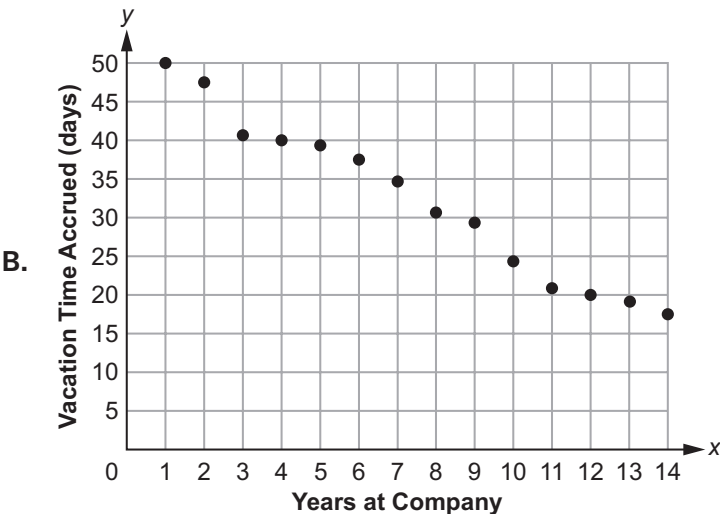
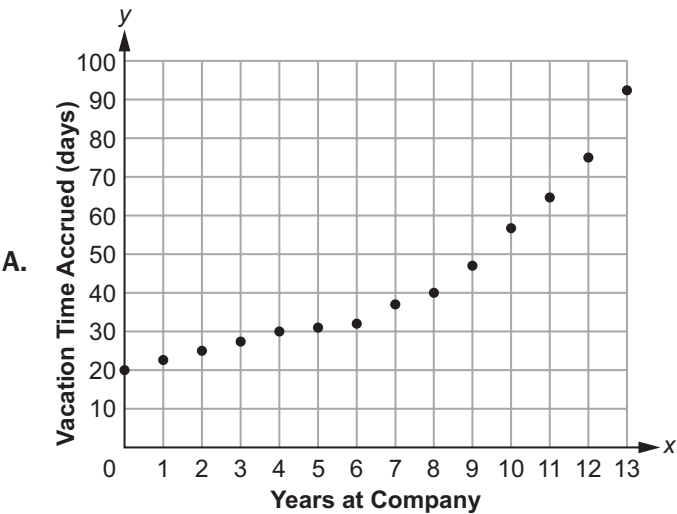
- Does it appear to be linear or exponential?
- Determine a function to describe the sequence.
- What would the 20th term of the sequence be?

Solution:

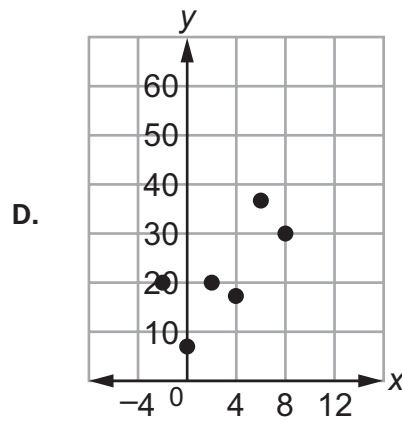
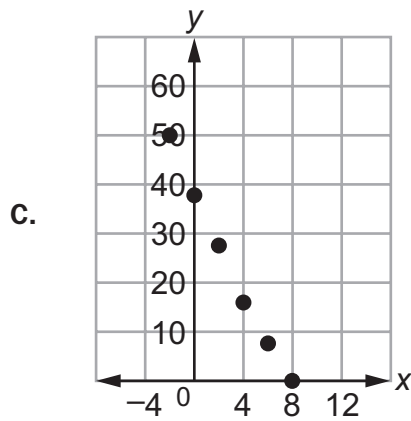
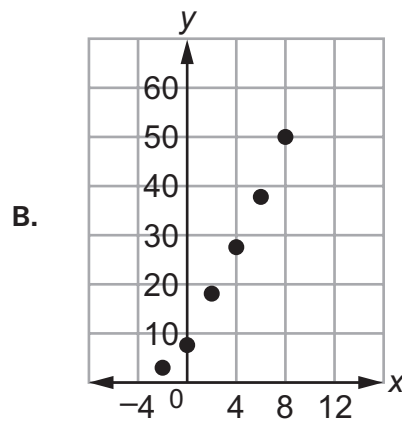
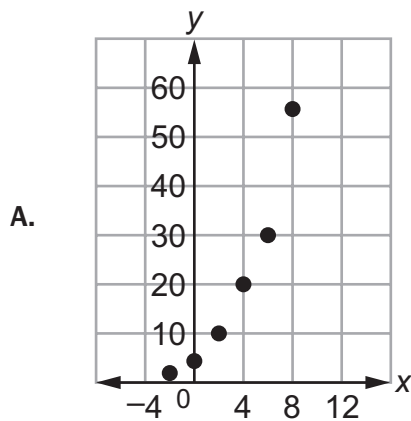
- Linear; the terms increase by a constant amount, 3.
- $f(x) = 3x + 4$. The growth rate is 3 and the first term is 4 more than 3 times 1. Notice the y-intercept is 4 and not 7, the initial term. The value 4 is when $x = 0$. Since 7 is when $x = 1$ and the growth rate is 3, the value for $x = 0$ is 3 less than 7, which is 4.
- 64; $f(20) = 3(20) + 4 = 64$

SAMPLE ITEMS

1. Which scatter plot BEST represents a model of linear growth?



2. Which scatter plot BEST represents a model of exponential growth?



3. Which table represents an exponential function?

A.

x	0	1	2	3	4
y	5	6	7	8	9

B.

x	0	1	2	3	4
y	22	44	66	88	110

C.

x	0	1	2	3	4
y	5	13	21	29	37

D.

x	0	1	2	3	4
y	3	9	27	81	243

Answers to Unit 3.7 Sample Items

1. C 2. A 3. D

3.8 Interpret Expressions for Functions in Terms of the Situation They Model

MGSE9-12.F.LE.5 Interpret the parameters in a linear ($f(x) = mx + b$) and exponential ($f(x) = a \cdot d^x$) function in terms of a context. (In the functions above, “ m ” and “ b ” are the parameters of the linear function, and “ a ” and “ d ” are the parameters of the exponential function.) In context, students should describe what these parameters mean in terms of change and starting value.

KEY IDEAS

A **parameter** is the independent variable or variables in a system of equations with more than one dependent variable. Though parameters may be expressed as letters when a relationship is generalized, they are not variables. A parameter as a constant term generally affects the intercepts of a function. If the parameter is a coefficient, in general it will affect the rate of change. Below are several examples of specific parameters.

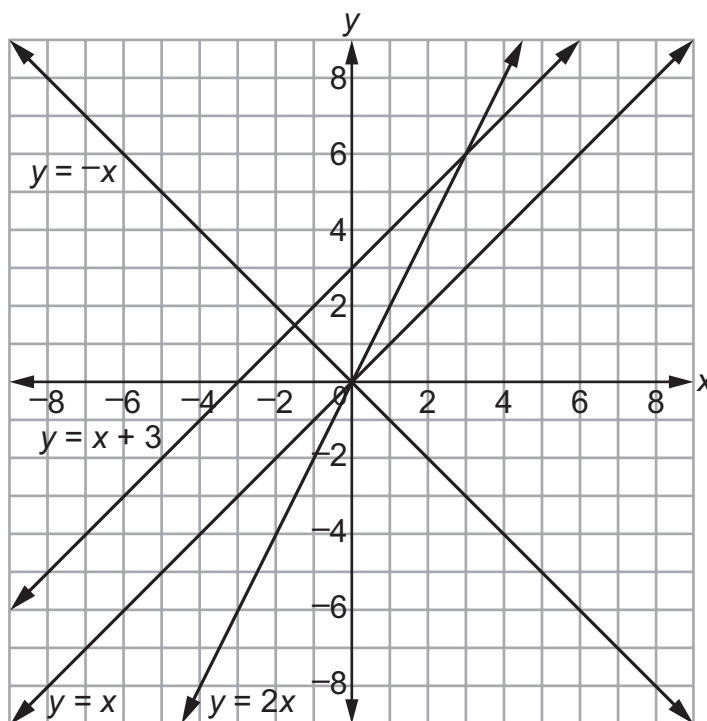
Equation	Parameter(s)
$y = 3x + 5$	coefficient 3, constant 5
$f(x) = \frac{9}{5}x + 32$	coefficient $\frac{9}{5}$, constant 32
$v(t) = v_0 + at$	coefficient a , constant v_0
$y = mx + b$	coefficient m , constant b

We can look at the effect of parameters on a linear function.

Example: Consider the lines $y = x$, $y = 2x$, $y = -x$, and $y = x + 3$. The coefficients of x are parameters. The $+3$ in the last equation is a parameter. We can make one table for all four lines and then compare their graphs.

x	y			
	$y = x$	$y = 2x$	$y = -x$	$y = x + 3$
-3	-3	-6	3	0
-2	-2	-4	2	1
-1	-1	-2	1	2
0	0	0	0	3
1	1	2	-1	4
2	2	4	-2	5
3	3	6	-3	6
4	4	8	-4	7

The four linear graphs show the effects of the parameters.



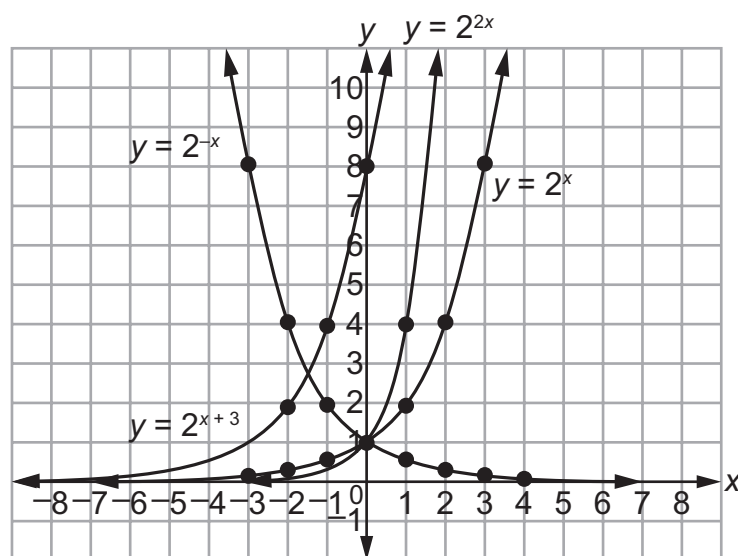
- Only $y = x + 3$ has a different y-intercept. The +3 translated the $y = x$ graph up 3 units.
- Both $y = x$ and $y = x + 3$ have the same slope (rate of change). The coefficients of the x-terms are both 1.
- The lines $y = -x$ and $y = 2x$ have different slopes than $y = x$. The coefficients of the x-terms, -1 and 2, affect the slopes of the lines.
- The line $y = -x$ is the reflection of $y = x$ over the x-axis. It is the only line with a negative slope.
- The rate of change of $y = 2x$ is twice that of $y = x$.

We can look at the effect of parameters on an exponential function, in particular, when applied to the independent variable, not the base.

Example: Consider the exponential curves $y = 2^x$, $y = 2^{-x}$, $y = 2^{2x}$, and $y = 2^{x+3}$. The coefficients of the exponent x are parameters. The +3 applied to the exponent x in the last equation is a parameter. We can make one table for all four exponentials and then compare the effects.

- $y = 2^{-x}$ is a mirror image of $y = 2^x$ with the y -axis as mirror. It has the same y -intercept.
- $y = 2^{2x}$ has the same y -intercept as $y = 2^x$ but rises much more steeply.
- $y = 2^{x+3}$ is the $y = 2^x$ curve translated 3 units to the left.

x	y			
	$y = 2^x$	$y = 2^{-x}$	$y = 2^{2x}$	$y = 2^{x+3}$
-3	$\frac{1}{8}$	8	$\frac{1}{64}$	1
-2	$\frac{1}{4}$	4	$\frac{1}{16}$	2
-1	$\frac{1}{2}$	2	$\frac{1}{4}$	4
0	1	1	1	8
1	2	$\frac{1}{2}$	4	16
2	4	$\frac{1}{4}$	16	32
3	8	$\frac{1}{8}$	64	64
4	16	$\frac{1}{16}$	256	128



Parameters show up in equations when there is a parent function. Parameters affect the shape and position of the parent function. When we determine a function that models a specific set of data, we are often called upon to find the parent function's parameters.

Example: Katherine has heard that you can estimate the outside temperature from the number of times a cricket chirps. It turns out that the warmer it is outside, the more a cricket will chirp. She has these three pieces of information:

- A cricket chirps 76 times a minute at 56° (76, 56).
- A cricket chirps 212 times per minute at 90° (212, 90).
- The relationship is linear.

Estimate the function.

The basic linear model or parent function is $f(x) = mx + b$, where m is the slope of the line and b is the y -intercept.

So, the slope, or rate of change, is one of our parameters. First we will determine the constant rate of change, called the slope, m .

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{90 - 56}{212 - 76} = \frac{34}{136} = \frac{1}{4}$$

We can create a function $T(c)$ to find the temperature when a cricket chirps c times using

$T(c) = \frac{1}{4}c + b$. Then we can substitute in one of our ordered pairs to determine b .

$$\begin{aligned}T(76) &= 56, \text{ so } \frac{1}{4}(76) + b = 56 \\19 + b &= 56 \\19 + b - 19 &= 56 - 19 \\b &= 37\end{aligned}$$

Our parameters are $m = \frac{1}{4}$ and $b = 37$.

Our function for the temperature is $T(c) = \frac{1}{4}c + 37$.

REVIEW EXAMPLES

- ◆ Alice finds that her flower bulbs multiply each year. She started with just 24 tulip plants. After one year she had 72 plants. Two years later she had 120. Find a linear function to model the growth of Alice's bulbs.

Year	0	1	2	3	4
Flower Bulbs	24	72	120	168	216

Solution:

The data points are (0, 24), (1, 72), and (2, 120). The linear model is $B(p) = m(p) + b$.

We know $b = 24$ because $B(0) = 24$ and $B(0)$ gives the vertical intercept.

$$\text{Find } m: m = \frac{120 - 72}{2 - 1} = \frac{48}{1} = 48.$$

The parameters are $m = 48$ and $b = 24$.

The function modeling the growth of the bulbs is $B(p) = 48p + 24$.

- ◆ Suppose Alice discovers she counted wrong the second year and she actually had 216 tulip plants. She realizes the growth is not linear because the rate of change was not the same. She must use an exponential model for the growth of her tulip bulbs. Find the exponential function to model the growth.

Solution:

We now have the points (0, 24), (1, 72), and (2, 216). We use a parent exponential model:

$$B(p) = a(b^p).$$

In the exponential model, the parameter a would be the initial number. So, $a = 24$. To find the base b , we substitute a coordinate pair into the parent function.

$$B(1) = 72, \text{ so } 24(b^1) = 72, b^1 = \frac{72}{24} = 3, \text{ so } b = 3.$$

Now we have the parameter and the base. The exponential model for Alice's bulbs would be

$$B(p) = 24(3^p).$$

SAMPLE ITEMS

1. If the parent function is $f(x) = mx + b$, what is the value of the parameter m for the line passing through the points $(-2, 7)$ and $(4, 3)$?

A. -9

B. $-\frac{3}{2}$

C. -2

D. $-\frac{2}{3}$

2. Consider this function for cell duplication. The cells duplicate every minute.

$$f(x) = 75(2)^x$$

- A. The 75 is the initial number of cells, and the 2 indicates that the number of cells doubles every minute.
- B. The 75 is the initial number of cells, and the 2 indicates that the number of cells increases by 2 every minute.
- C. The 75 is the number of cells at 1 minute, and the 2 indicates that the number of cells doubles every minute.
- D. The 75 is the number of cells at 1 minute, and the 2 indicates that the number of cells increases by 2 every minute.

Answers to Unit 3.8 Sample Items

1. D 2. A

UNIT 4: DESCRIBING DATA

In this unit, students will learn informative ways to display both categorical and quantitative data. They will learn ways of interpreting those displays and pitfalls to avoid when presented with data. Students will learn how to determine the mean absolute deviation. Among the methods they will study are two-way frequency charts for categorical data and lines of best fit for quantitative data. Measures of central tendency will be revisited along with measures of spread.

4.1 Summarize, Represent, and Interpret Data on a Single Count or Measurable Variable

MGSE9-12.S.ID.1 Represent data with plots on the real number line (dot plots, histograms, and box plots).

MGSE9-12.S.ID.2 Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, mean absolute deviation) of two or more different data sets.

MGSE9-12.S.ID.3 Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).

KEY IDEAS

Two **measures of central tendency** that help describe a data set are mean and median.

- The **mean** is the sum of the data values divided by the total number of data values.
- The **median** is the middle value when the data values are written in order from least to greatest. If a data set has an even number of data values, the median is the mean of the two middle values.

The **first quartile**, or the **lower quartile**, Q_1 , is the median of the lower half of a data set.

Example: Ray's scores on his mathematics tests were 70, 85, 78, 90, 84, 82, and 83. To find the first quartile of his scores, write them in order from least to greatest:

70, 78, 82, 83, 84, 85, 90

The scores in the lower half of the data set are 70, 78, and 82. The median of the lower half of the scores is 78.

So, the first quartile is 78.

The **third quartile**, or the **upper quartile**, Q_3 , is the median of the upper half of a data set.

Example: Referring to the previous example, the upper half of Ray's scores is 84, 85, and 90. The median of the upper half of the scores is 85.

So, the third quartile is 85.

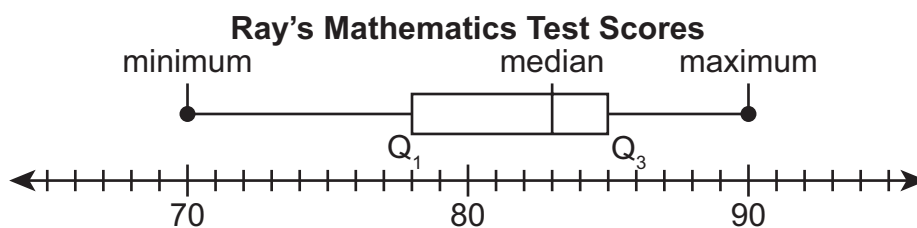
There is a review example in this section that describes how to find Q_1 and Q_3 where you must average values to find the median of the upper and lower quartiles.

The **interquartile range (IQR)** of a data set is the difference between the third and first quartiles, or $Q_3 - Q_1$.

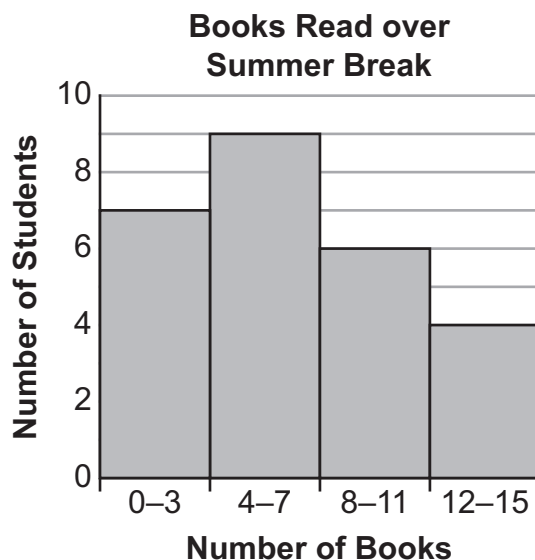
Example: Referring again to the example of Ray's scores, to find the interquartile range, subtract the first quartile from the third quartile. The interquartile range of Ray's scores is $85 - 78 = 7$.

The most common displays for quantitative data are dot plots, histograms, box plots, and frequency distributions. A **box plot** is a diagram used to display a data set that uses quartiles to form the center box and the minimum and maximum to form the whiskers.

Example: For the data about Ray's mathematics scores, the box plot would look like the one shown below:



A **histogram** is a graphical display that subdivides the data into class intervals, called **bins**, and uses a rectangle to show the frequency of observations in those intervals—for example, you might use intervals of 0–3, 4–7, 8–11, and 12–15 for the number of books students read over summer break.



Sometimes, distributions are characterized by extreme values that differ greatly from the other observations. These extreme values are called **outliers**. A data value is an outlier if it is less than $Q_1 - 1.5 \cdot \text{IQR}$ or above $Q_3 + 1.5 \cdot \text{IQR}$.

Example: This example shows the effect that an outlier can have on a measure of central tendency.

The mean is one of several measures of central tendency that can be used to describe a data set. The main limitation of the mean is that, because every data value directly affects the result, it can be affected greatly by outliers. Consider these two sets of quiz scores:

Student P: {8, 9, 9, 9, 10}

Student Q: {3, 9, 9, 9, 10}

Both students consistently performed well on quizzes and both have the same median and mode score, 9. Student Q, however, has a mean quiz score of 8, while student P has a mean quiz score of 9. Although many instructors accept the use of a mean as being fair and representative of a student's overall performance in the context of test or quiz scores, it can be misleading because it fails to describe the variation in a student's scores, and the effect of a single score on the mean can be disproportionately large, especially when the number of scores is small.

Mean Absolute Deviation is the distance each data value is from the mean of the data set. This helps to get a sense of how spread out a data set is.

Example: This example shows two sets of data that have the same mean but different mean absolute deviations. Consider the quiz scores of two students:

Student R: {3, 6, 8, 8, 9, 10, 12}

Student S: {1, 1, 3, 7, 14, 15, 15}

The mean score of student R is 8, and the mean score of student S is also 8. Determining the mean does not provide us with which student was more consistent. Which student is more consistent is what the mean absolute deviation will provide. We can use this formula:

$$\frac{\sum_{i=1}^n |x_i - \bar{x}|}{n}$$

to find the mean absolute deviation. Remember that x_i is the i th data value, \bar{x} is the mean of the data, and n is the sample size. To apply the formula, we need to find the sum of the difference of the terms and

the mean $\sum_{i=1}^n |x_i - \bar{x}|$. So,

$$\text{Student R: } |3 - 8| + |6 - 8| + |8 - 8| + |8 - 8| + |9 - 8| + |10 - 8| + |12 - 8| = 14$$

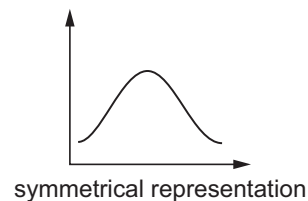
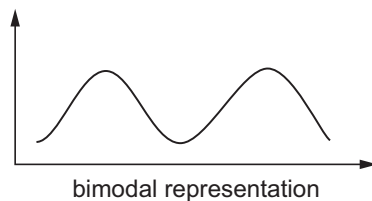
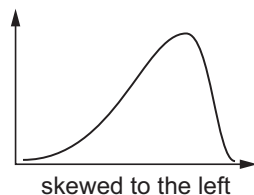
$$\text{Student S: } |1 - 8| + |1 - 8| + |3 - 8| + |7 - 8| + |14 - 8| + |15 - 8| + |15 - 8| = 40$$

The final step is to divide the sums by the number of data, n .


$$\text{Student R: } \frac{14}{8} = 1.75 \quad \text{Student S: } \frac{40}{8} = 5$$

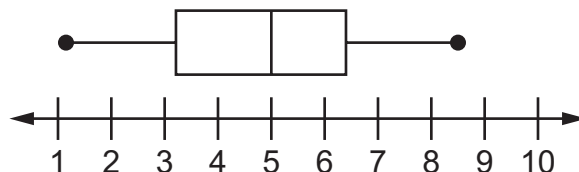
Since the mean absolute deviation of student R is smaller than student S, this means the quiz scores of student R were more consistent between all 8 quizzes.

Skewness refers to the type and degree of a distribution's asymmetry. A distribution is skewed to the left if it has a longer tail on the left side. If a distribution has a longer tail on the right, it is skewed to the right. Generally, distributions have only one peak, but there are distributions called **bimodal** or **multimodal** where there are two or more peaks, respectively. A box plot can present a fair representation of a data set's distribution.



Important Tip

-  The extent to which a data set is distributed normally can be determined by observing its skewness. Most of the data should lie in the middle near the median. The mean and the median should be fairly close. The left and right tails of the distribution curve should taper off. There should be only one peak, and it should neither be too high nor too flat.

**REVIEW EXAMPLES**

- ◆ Josh and Richard each earn tips at their part-time jobs. This table shows their earnings from tips for five days.

Total Tips by Day

Day	Josh's Tips	Richard's Tips
Monday	\$40	\$40
Tuesday	\$20	\$45
Wednesday	\$36	\$53
Thursday	\$28	\$41
Friday	\$31	\$28

- Who had the greater median earnings from tips? What is the difference in the median of Josh's earnings from tips and the median of Richard's earnings from tips?
- What is the difference in the interquartile range for Josh's earnings from tips and the interquartile range for Richard's earnings from tips?

Solution:

- Write Josh's earnings from tips in order from the least to greatest. Then, identify the middle value.
\$20, \$28, **\$31**, \$36, \$40

Josh's median earnings from tips is \$31.

Write Richard's earnings from tips in order from the least to greatest. Then, identify the middle value.

\$28, \$40, **\$41**, \$45, \$53

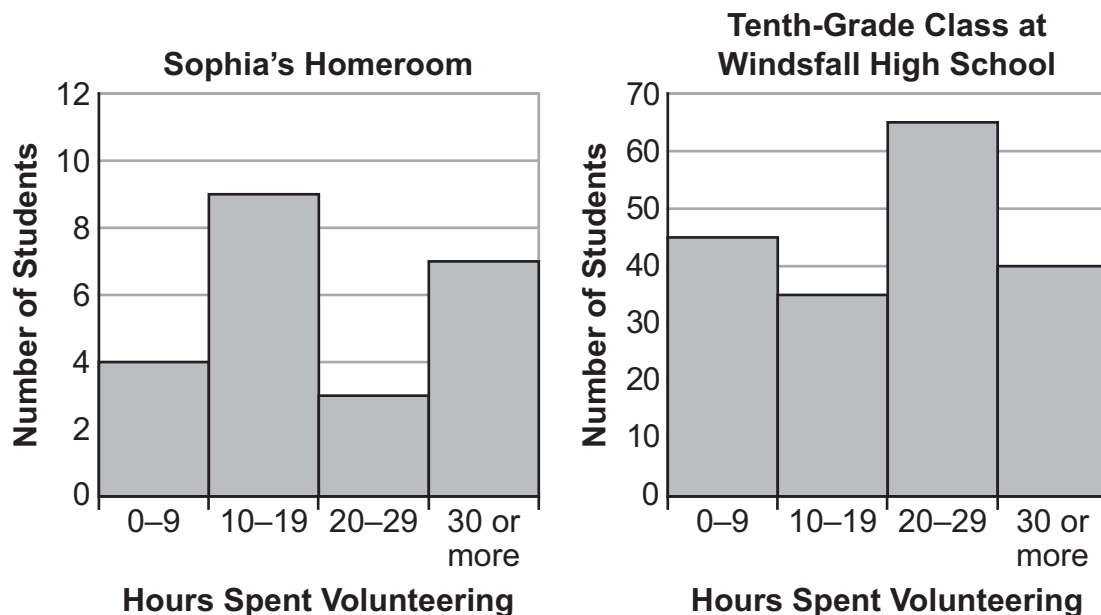
Richard had the greater median earnings from tips. The difference in the median of the earnings from tips is $\$41 - \$31 = \$10$.

- b. Since there is an even number of values in each half of data, the lower and upper quartile are found by finding the median of the sets, which means averaging the values. For Josh's earnings from tips, the lower quartile is \$24 and the upper quartile is \$38. The interquartile range is $\$38 - \24 , or \$14.

For Richard's earnings from tips, the lower quartile is \$34 and the upper quartile is \$49. The interquartile range is $\$49 - \34 , or \$15.

The difference in Josh's interquartile range and Richard's interquartile range is $\$15 - \14 , or \$1.

- ◆ Sophia is a student at Windsfall High School. These histograms give information about the number of hours spent volunteering by each of the students in Sophia's homeroom and by each of the students in the tenth-grade class at her school.



- Compare the lower quartiles of the data in the histograms.
- Compare the upper quartiles of the data in the histograms.
- Compare the medians of the data in the histograms.

Solution:

- You can add the number of students given by the height of each bar to find that there are 23 students in Sophia's homeroom. The lower quartile is the median of the first half of the data. That would be found within the 10–19 hours interval.

You can add the number of students given by the height of each bar to find that there are 185 students in the tenth-grade class. The lower quartile for this group is found within the 10–19 hours interval.

The interval of the lower quartile of the number of hours spent volunteering by each student in Sophia's homeroom is the same as the interval of the lower quartile of the number of hours spent volunteering by each student in the tenth-grade class.

- b. The upper quartile is the median of the second half of the data. For Sophia's homeroom, that would be found in the 30 or greater interval.

For the tenth-grade class, the upper quartile is found within the 20–29 hours interval.

The upper quartile of the number of hours spent volunteering by each student in Sophia's homeroom is greater than the upper quartile of the number of hours spent volunteering by each student in the tenth-grade class.

- c. The median is the middle data value in a data set when the data values are written in order from least to greatest. The median for Sophia's homeroom is found within the 10–19 hours interval.

The median for the tenth-grade class is found within the 20–29 hours interval.

The median of the number of hours spent volunteering by each student in Sophia's homeroom is less than the median of the number of hours spent volunteering by each student in the tenth-grade class.

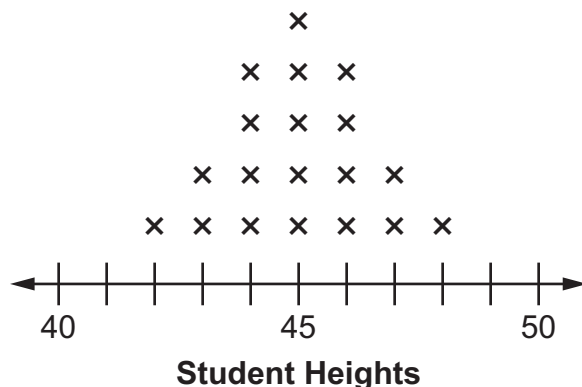
- ◆ Mr. Storer, the physical education teacher at an elementary school, measured and rounded, to the nearest whole inch, the height of each of his students. He organized his data in this chart.

Height (inches)	Frequency
42	1
43	2
44	4
45	5
46	4
47	2
48	1

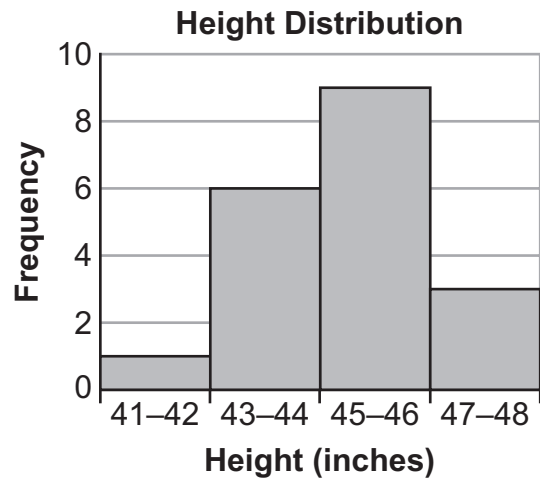
- Make a dot plot for the data.
- Make a histogram for the data with 4 classes.
- Make a box plot for the data.

Solution:

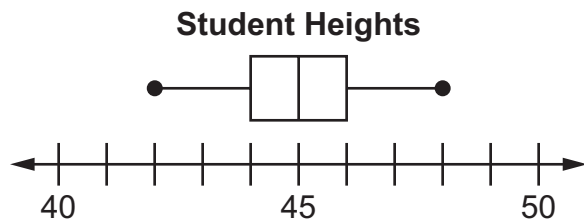
- a.



b.

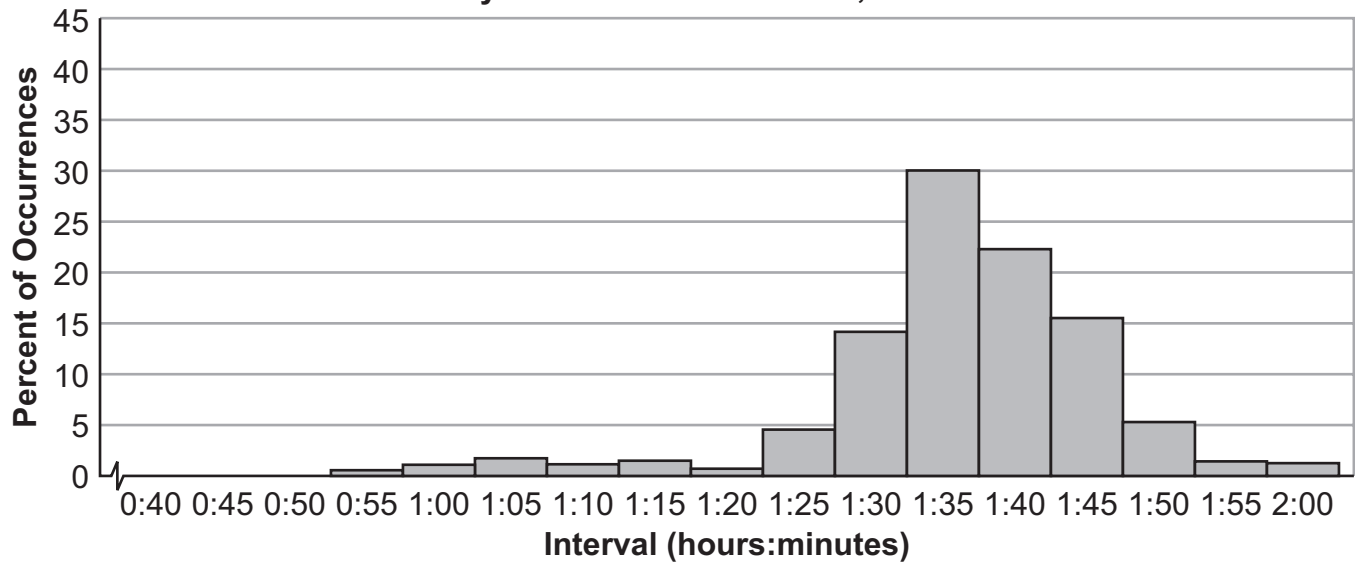


c.

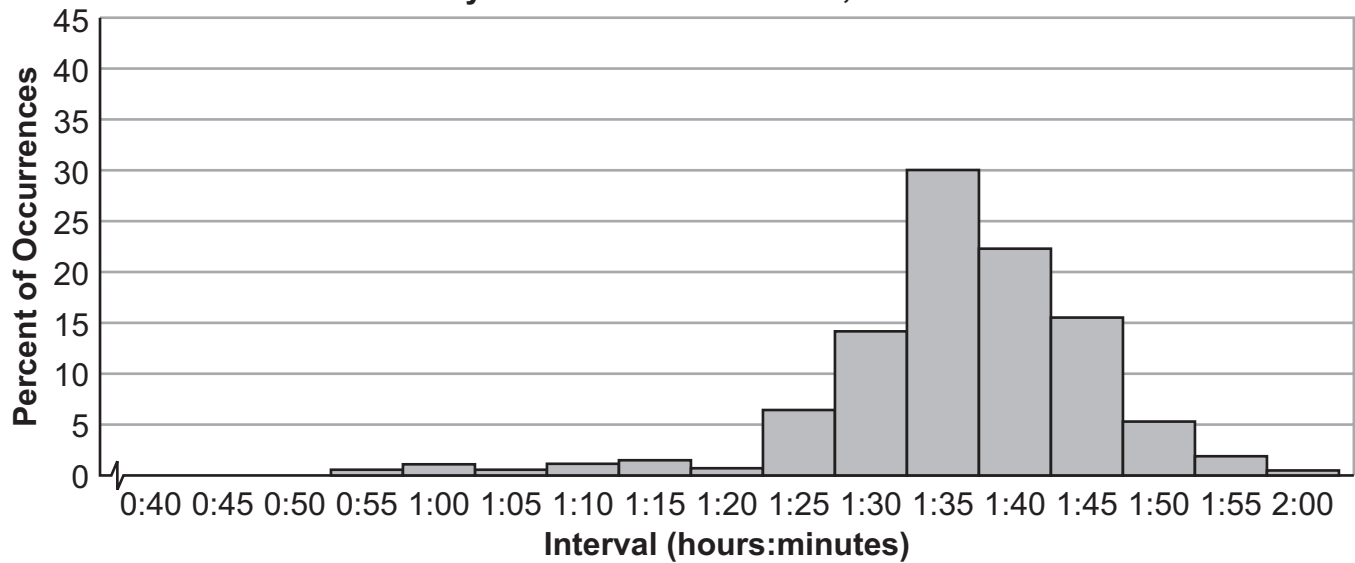


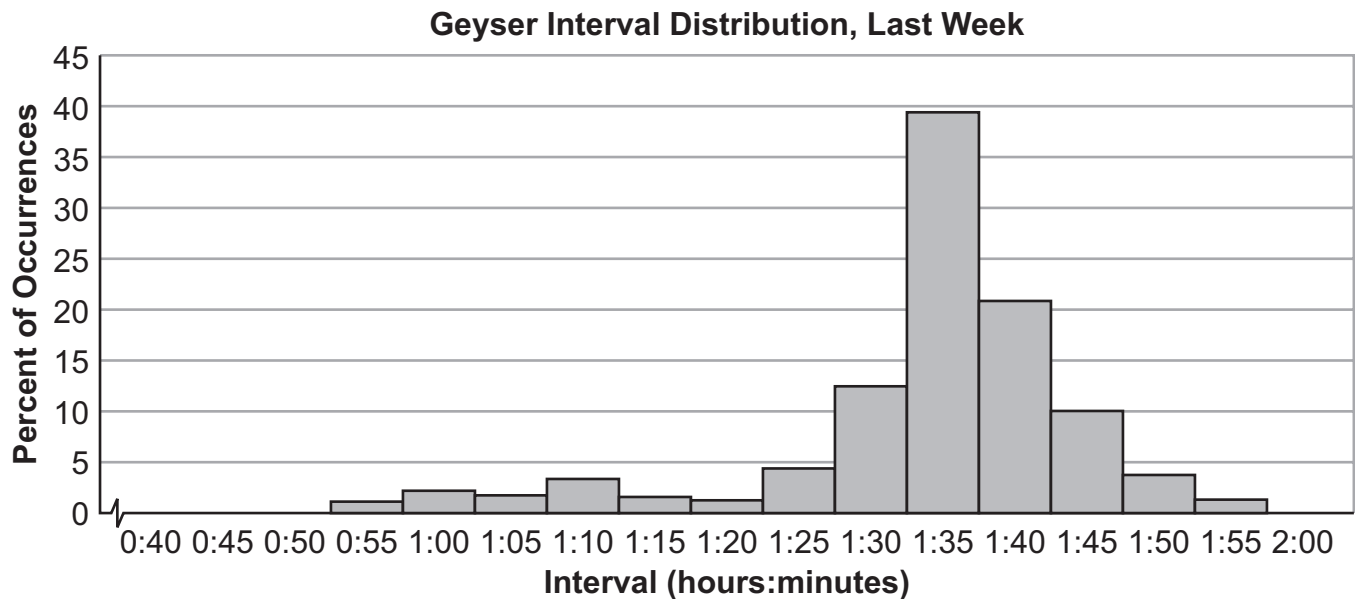
- ◆ A geyser in a national park erupts fairly regularly. In more recent times, it has become less predictable. It was observed last year that the time interval between eruptions was related to the duration of the most recent eruption. The distribution of its interval times for last year is shown in the following graphs.

Geyser Interval Distribution, Last Year



Geyser Interval Distribution, Last Month





- Does the Last Year distribution seem skewed or uniform?
- Compare Last Week's distribution to Last Month's distribution.
- What does the Last Year distribution tell you about the interval of time between the geyser's eruptions?

Solution:

- The Last Year distribution appears to be skewed to the left (negative). Most of the intervals approach 90 minutes.
- Last Week's distribution seems more skewed to the left than Last Month's. It is also more asymmetric because of its high number of 1-hour-and-35-minute intervals between eruptions. Last Month's distribution appears to have the highest percentage of intervals longer than 1 hour 30 minutes between eruptions.
- The Last Year distribution shows that the geyser rarely erupts an hour after its previous eruption. Most visitors will have to wait more than 90 minutes to see two eruptions.

SAMPLE ITEMS

1. This table shows the average low temperature, in °F, recorded in Macon, GA, and Charlotte, NC, over a six-day period.

Day	1	2	3	4	5	6
Temperature in Macon, GA (°F)	71	72	66	69	71	73
Temperature in Charlotte, NC (°F)	69	64	68	74	71	75

Which conclusion can be drawn from the data?

- A. The interquartile range of the temperatures is the same for both cities.
- B. The lower quartile for the temperatures in Macon is less than the lower quartile for the temperatures in Charlotte.
- C. The mean and median temperatures in Macon were higher than the mean and median temperatures in Charlotte.
- D. The upper quartile for the temperatures in Charlotte was less than the upper quartile for the temperatures in Macon.
2. A school was having a coat drive for a local shelter. The amount of coats each homeroom collected for the freshman and sophomore classes are shown.
- Freshman Homerooms: 4, 8, 6, 8, 7, 3
 - Sophomore Homerooms: 6, 9, 3, 6, 11, 7

Which statement is true about the average number of coats collected by the freshman and sophomore homerooms?

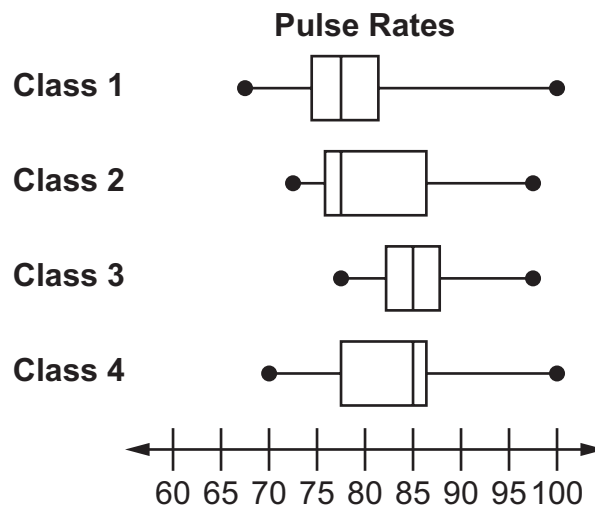
- A. The freshmen averaged 1 more coat collected than the sophomores.
- B. The freshmen averaged the same number of coats collected as the sophomores.
- C. The sophomores averaged 1 more coat collected than the freshmen.
- D. The sophomores averaged 3 more coats collected than the freshmen.

3. A reading teacher recorded the number of pages read in an hour by each of her students. The numbers are shown below.

44, 49, 39, 43, 50, 44, 45, 49, 51

For this data, which summary statistic is NOT correct?

- A. The minimum is 39.
 - B. The lower quartile is 44.
 - C. The median is 45.
 - D. The maximum is 51.
4. A science teacher recorded the pulse of each of the students in her classes after the students had climbed a set of stairs. She displayed the results, by class, using the box plots shown.



Which class generally had the highest pulse after climbing the stairs?

- A. Class 1
- B. Class 2
- C. Class 3
- D. Class 4

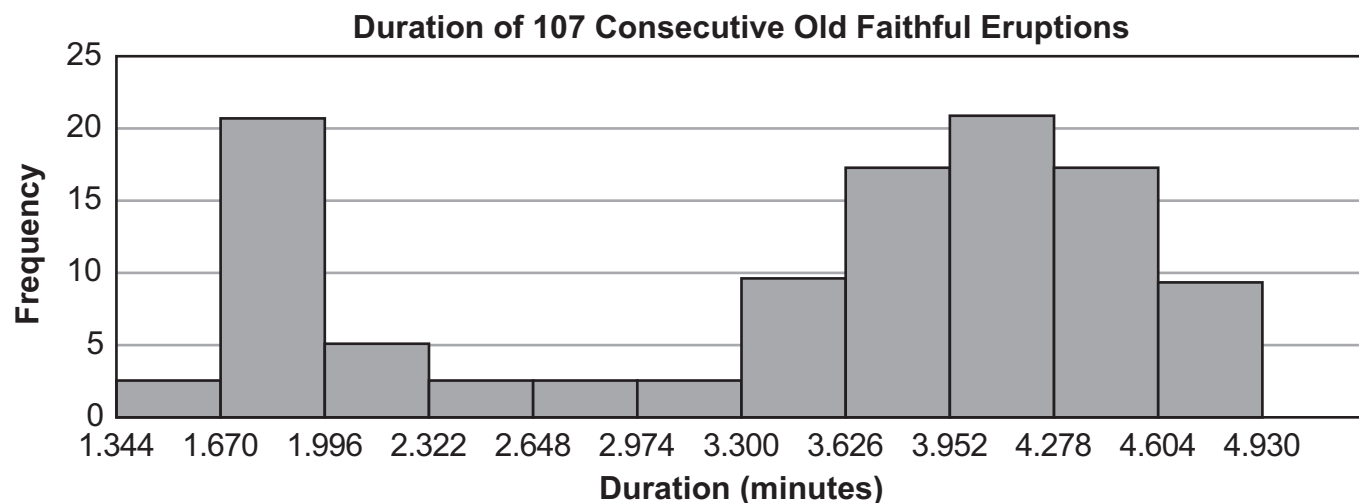
5. Peter went bowling, Monday to Friday, two weeks in a row. He only bowled one game each time he went. He kept track of his scores below.

Week 1: 70, 70, 70, 73, 75

Week 2: 72, 64, 73, 73, 75

What is the BEST explanation for why Peter's Week 2 mean score was lower than his Week 1 mean score?

- A. Peter received the same score three times in Week 1.
B. Peter had one very low score in Week 2.
C. Peter did not beat his high score from Week 1 in Week 2.
D. Peter had one very high score in Week 1.
6. This histogram shows the frequency distribution of duration times for 107 consecutive eruptions of the Old Faithful geyser. The duration of an eruption is the length of time, in minutes, from the beginning of the spewing of water until it stops. What is the BEST description for the distribution?



- A. bimodal
B. uniform
C. multiple outlier
D. skewed to the right

Answers to Unit 4.1 Sample Items

1. C 2. C 3. B 4. C 5. B 6. A

4.2 Summarize, Represent, and Interpret Data on Two Categorical and Quantitative Variables

MGSE9-12.S.ID.5 Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.

MGSE9-12.S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

MGSE9-12.S.ID.6a Decide which type of function is most appropriate by observing graphed data, charted data, or by analysis of context to generate a viable (rough) function of best fit. Use this function to solve problems in context. Emphasize linear, quadratic, and exponential models.

MGSE9-12.S.ID.6c Using given or collected bivariate data, fit a linear function for a scatter plot that suggests a linear association.

KEY IDEAS

There are essentially two types of data: **categorical** and **quantitative**. Examples of categorical data are color, type of pet, gender, ethnic group, religious affiliation, etc. Examples of quantitative data are age, years of schooling, height, weight, test score, etc. Researchers use both types of data but in different ways. Bar graphs and pie charts are frequently associated with categorical data. Box plots, dot plots, and histograms are used with quantitative data. The measures of central tendency (mean, median, and mode) apply to quantitative data. Frequencies can apply to both categorical and quantitative data.

Bivariate data consist of pairs of linked numerical observations, or frequencies of things in categories. Numerical bivariate data can be presented as ordered pairs and in any way that ordered pairs can be presented: as a set of ordered pairs, as a table of values, or as a graph on the coordinate plane.

Categorical example: frequencies of gender and club memberships for 9th graders

A bivariate chart, or **two-way frequency chart**, is often used with data from two categories. Each category is considered a variable, and the categories serve as labels in the chart. Two-way frequency charts are made of cells. The number in each cell is the frequency of things that fit both the row and column categories for the cell. From the two-way frequency chart that follows, we see that there are 12 males in the band and 3 females in the chess club.

Participation in School Activities			
School Club	Gender		
	Male	Female	Totals
Band	12	21	33
Chorus	15	17	32
Chess	16	3	19
Latin	7	9	16
Yearbook	28	7	35
Totals	78	57	135

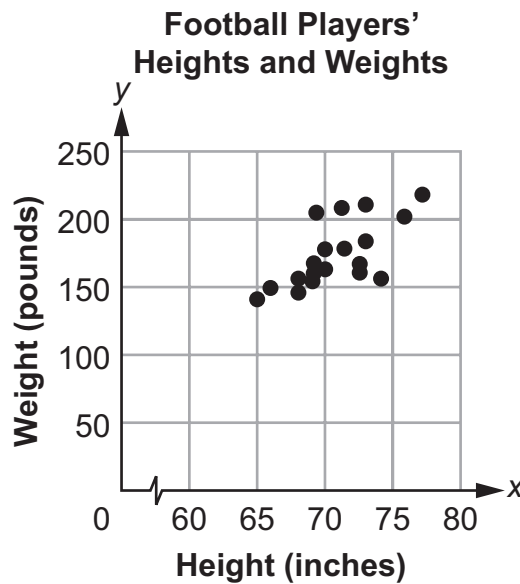
If no person or thing can be in more than one category per scale, the entries in each cell are called **joint frequencies**. The frequencies in the cells and the totals tell us about the percentages of students engaged in different activities based on gender. For example, we can determine from the chart that if we picked at random from the students, we are least likely to find a female in the chess club because only 3 of 135 students are females in the chess club. The most popular club is yearbook, with 35 of 135 students in that club. The values in the table can be converted to percentages, which will give us an idea of the composition of each club by gender. We see that close to 14% of the students are in the chess club, and there are more than five times as many males as females.

Participation in School Activities			
School Club	Gender		
	Male	Female	Totals
Band	8.9%	15.5%	24.4%
Chorus	11.1%	12.6%	23.7%
Chess	11.9%	2.2%	14.1%
Latin	5.2%	6.7%	11.9%
Yearbook	20.7%	5.2%	25.9%
Totals	57.8%	42.2%	100%

There are also what we call **marginal frequencies** in the bottom and right margins (the shaded cells in the table). These frequencies lack one of the categories. For our example, the frequencies at the bottom represent percentages of males and females in the school population. The marginal frequencies on the right represent percentages of club membership.

Lastly, also associated with two-way frequency charts are **conditional frequencies**. These are not usually in the body of the chart but can be readily calculated from the cell contents. One conditional frequency would be the percentage of females in school who are in chorus. The working condition is that the person is female. If 12.6% of the entire school population is made up of females in the chorus and 42.2% of the student body is made up of females, then $12.6\% / 42.2\%$, or 29.9%, of the females in the school are in the chorus (also, 17 of 57 females).

Quantitative example: Consider this chart of heights and weights of players on a football team



A scatter plot is often used to present bivariate quantitative data. Each variable is represented on an axis, and the axes are labeled accordingly. Each point represents a player's height and weight. For example, one of the points represents a height of 66 inches and weight of 150 pounds. The scatter plot shows two players that are 70 inches tall because there are two dots for that height.

A **scatter plot** displays data as points on a grid using the associated numbers as coordinates. The way the points are arranged by themselves in a scatter plot may or may not suggest a relationship between the two variables. In the scatter plot about the football players shown, it appears there may be a relationship between height and weight because as the players get taller, they seem to generally increase in weight; that is, the points are positioned higher as you move to the right. Bivariate data may have an underlying relationship that can be modeled by a mathematical function. For the purposes of this unit we will consider linear models.

Example: Melissa would like to determine whether there is a relationship between study time and mean test score. She recorded the mean study time per test and the mean test score for students in three different classes.

These are the data for Class 1.

Class 1 Test Score Analysis	
Mean Study Time (hours)	Mean Test Score
0.5	63
1	67
1.5	72
2	76
2.5	80
3	85
3.5	89

Notice that, for these data, as the mean study time increases, the mean test score increases. It is important to consider the **rate of increase** when deciding which algebraic model to use. In this case, the mean test score increases by approximately 4 points for each 0.5-hour increase in mean study time. When the rate of increase is close to constant, as it is here, the best model is most likely a linear function.

This next table shows Melissa's data for Class 2.

Class 2 Test Score Analysis	
Mean Study Time (hours)	Mean Test Score
0.5	60
1	61
1.5	63
2	68
2.5	74
3	82
3.5	93

In these data as well, the mean test score increases as the mean study time increases. However, the rate of increase is not constant. The differences between each successive mean test score are 1, 2, 5, 6, 8, and 11.

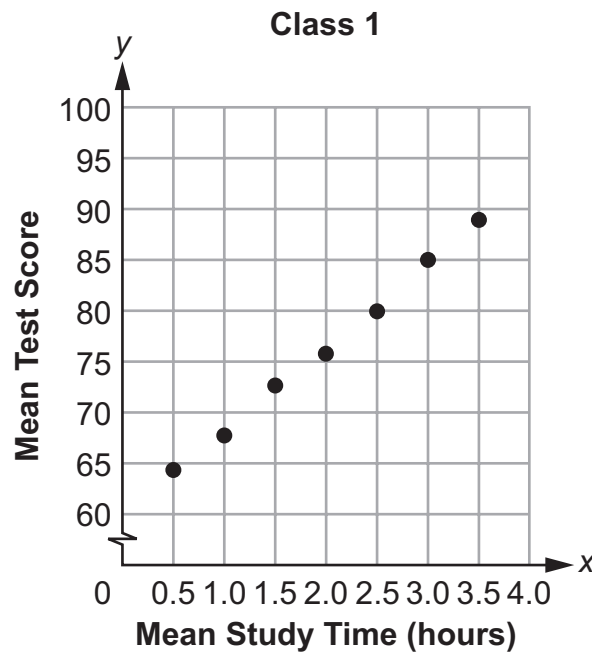
This table shows Melissa's data for Class 3.

Class 3 Test Score Analysis	
Mean Study Time (hours)	Mean Test Score
0.5	71
1	94
1.5	87
2	98
2.5	69
3	78
3.5	91

In these data, as the mean study time increases, there is no consistent pattern in the mean test score. As a result, there does not appear to be any clear relationship between the mean study time and mean test score for this particular class.

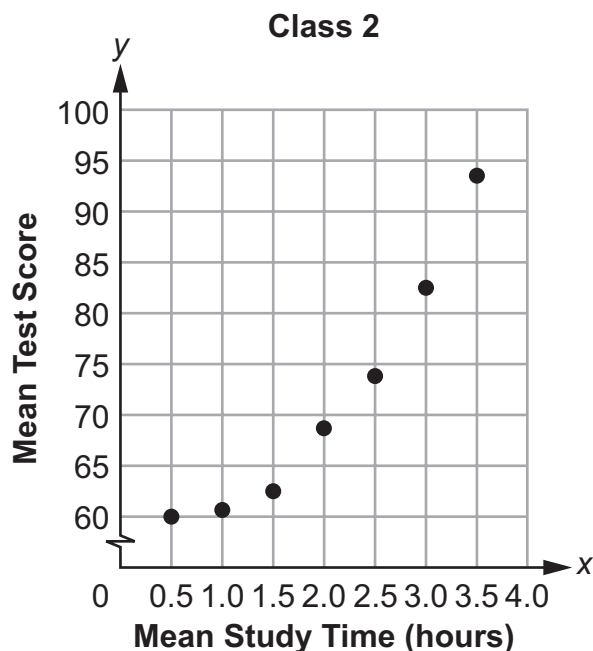
Often, patterns in bivariate data are more easily seen when the data are plotted on a coordinate grid.

Example: This graph shows Melissa's data for Class 1.



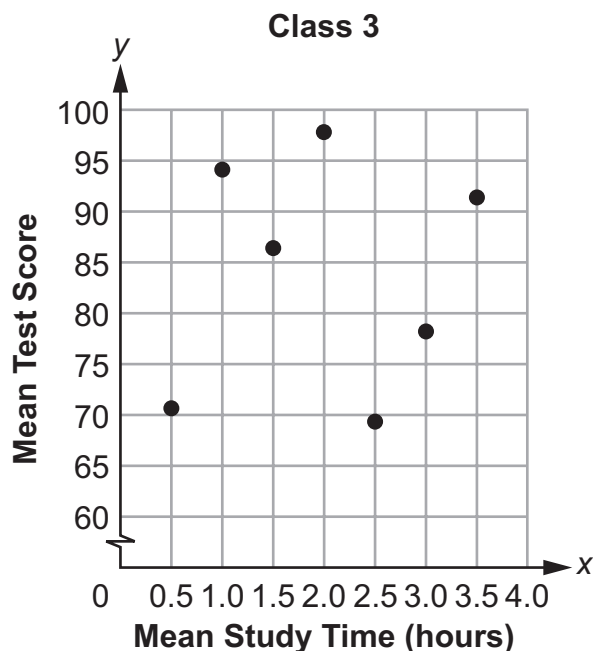
In this graph, the data points are all very close to being on the same line. This is further confirmation that a linear model is appropriate for this class.

This graph shows Melissa's data for Class 2.



In this graph, the data points appear to lie on a curve, rather than on a line, with a rate of increase that increases as the value of x increases. It appears that an exponential model may be more appropriate than a linear model for these data.

This graph shows Melissa's data for Class 3.



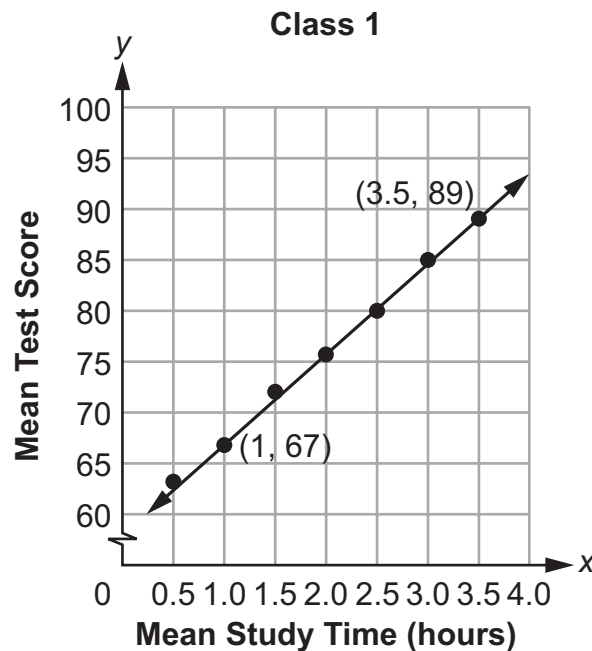
In this graph, the data points do not appear to lie on a line or on a curve. Neither a linear model nor an exponential model is appropriate to represent the data.

A **line of best fit** (trend or regression line) is a straight line that best represents the data on a scatter plot. This line may pass through some of the points, none of the points, or all of the points. In the previous examples, only the Class 1 scatter plot looks like a linear model would be a good fit for the points. In the other classes, a curved graph would seem to pass through more of the points. For Class 2, perhaps an exponential model would produce a better-fitting curve. Since Class 3 appears to have no correlation, creating a model may not produce the desired results.

When a linear model is indicated, there are several ways to find a function that approximates the y -value for any given x -value. A method called **regression** is the best way to find a line of best fit, but it requires extensive computations and is generally done on a computer or graphing calculator.

Example: The graph shows Melissa's data for Class 1 with a line of best fit drawn. The equation of the line can be determined by using technology to enter the data and then using the linear regression feature of your technology. You will get values for m and b for the equation $y = mx + b$. The equation for the data is $y = 8.7x + 58.6$.

Determining the line of best fit without the use of technology will lead to many different equations depending on the two points chosen to construct the line. Make a scatter plot for the given data. Draw a straight line that best represents the data of your scatter plot. Make sure to extend your line so that it is near or intersects the y -axis. Next you will need to choose any two points that fall on or fall closest to the straight line you drew for your scatter plot. Then you will determine m , the slope of those two points for the equation $y = mx + b$.



Notice that five of the seven data points are on the line. This represents a very strong positive relationship for study time and test scores since the line of best fit is positive and a very tight fit to the data points.

We have chosen the points (3.5, 89) and (1, 67). Other points may be chosen. Then we determine the slope of the line that passes through the two points.

$$m = \frac{67 - 89}{1 - 3.5} = \frac{-22}{-2.5} = 8.8$$

Then write the equation of the line as shown.

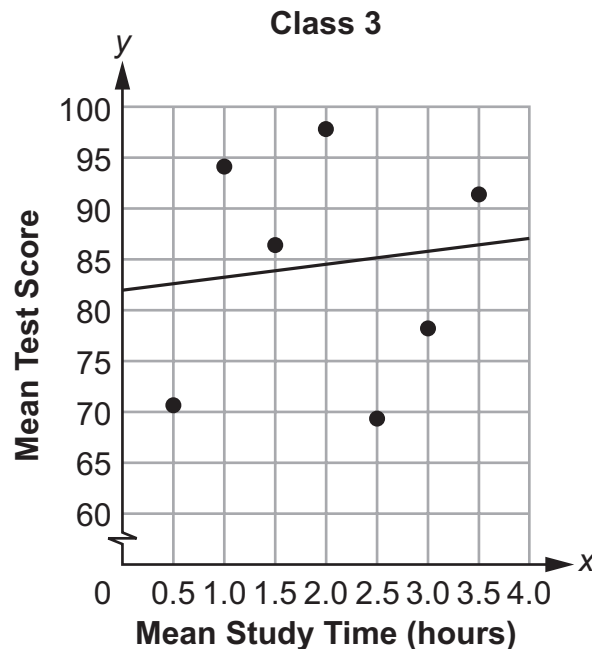
$$y - y_1 = m(x - x_1)$$

$$y - 89 = 8.8(x - 3.5)$$

$$y - 89 = 8.8x - 30.8$$

$$y = 8.8x + 58.2$$

This next graph shows Melissa's data for Class 3 with a line of best fit added. The equation of the line is $y = 0.8x + 83.1$.

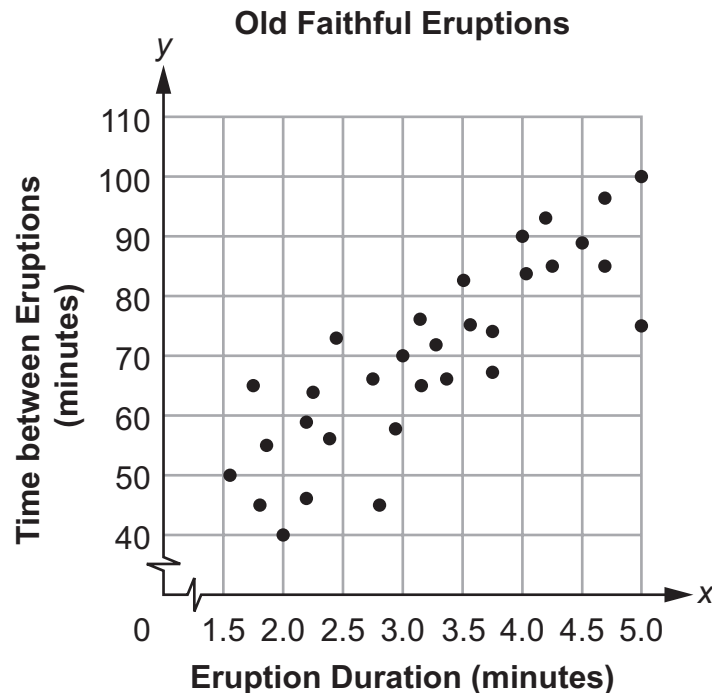


Although a line of best fit can be calculated for this set of data, notice that most of the data points are not very close to the line. In this case, although there is some correlation between study time and test scores, the amount of correlation is very small.

This is called the correlation coefficient, which is discussed in more detail in the next section about linear models.

REVIEW EXAMPLES

- ◆ Barbara is considering visiting Yellowstone National Park. She has heard about Old Faithful, a geyser, and she wants to make sure she sees it erupt. At one time, it erupted just about every hour. That is not the case today. The time between eruptions varies. Barbara went on the Web and found a scatter plot of how long an eruption lasted compared to the wait time between eruptions. She learned that, in general, the longer the wait time, the longer the eruption lasts. The eruptions take place anywhere from 45 minutes to 125 minutes apart. They currently average 90 minutes apart.



- For an eruption that lasts 4 minutes, about how long would the wait time be for the next eruption?
- What is the shortest duration time for an eruption?
- Determine whether the scatter plot has a positive or a negative correlation, and explain how you know.

Solution:

- After a 4-minute eruption, it would be between 80 and 90 minutes for the next eruption.
- The shortest eruptions appear to be a little more than 1.5 minutes (1 minute and 35 seconds).
- The scatter plot has a positive correlation because as the eruption duration increases, the time between eruptions increases.

- ◆ The environment club is interested in the relationship between x , the number of canned beverages sold in the cafeteria, and y , the number of cans that are recycled. The data they collect are listed in this chart.

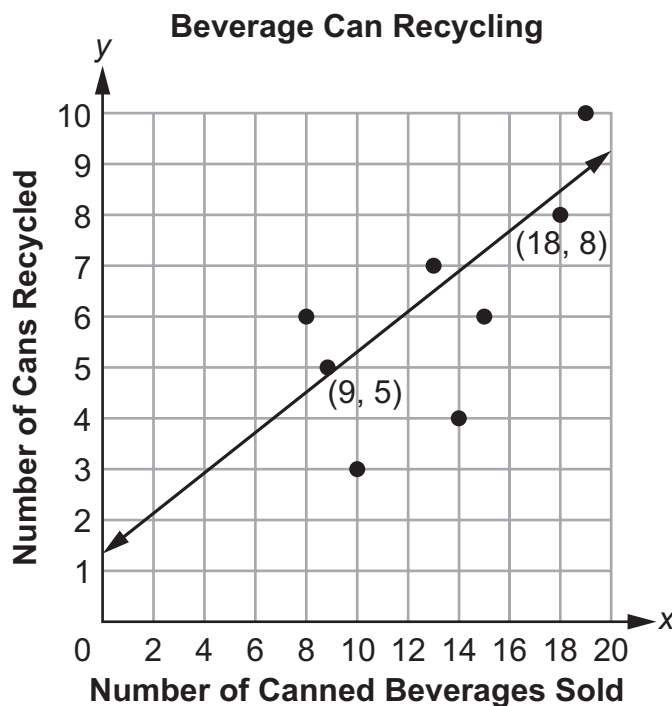
Beverage Can Recycling

Number of Canned Beverages Sold	18	15	19	8	10	13	9	14
Number of Cans Recycled	8	6	10	6	3	7	5	4

Determine an equation of a line of best fit for the data.

Solution:

Remember, determining the line of best fit without the use of technology will lead to many different equations depending on the two points chosen to construct the line. Make a scatter plot for the given data. Draw a straight line that best represents the data of your scatter plot. Make sure to extend your line so that it is near or intersects the y -axis.



Next you will need to choose any two points that fall on or fall closest to the straight line you drew for your scatter plot. Then you will determine m , the slope of those two points for the equation $y = mx + b$. We have chosen the points (18, 8) and (9, 5). Other points may be chosen. Then we determine the slope of the line that passes through the two points.

$$m = \frac{5 - 8}{9 - 18} = \frac{-3}{-9} = \frac{1}{3}$$

Then write the equation of the line as shown.

$$y - y_1 = m(x - x_1)$$

$$y - 8 = \frac{1}{3}(x - 18)$$

$$y - 8 = \frac{1}{3}x - 6$$

$$y = \frac{1}{3}x + 2$$

- ◆ A fast-food restaurant wants to determine whether the season of the year affects the choice of soft-drink size purchased. It surveyed 278 customers, and the table shows the results. The drink sizes were small, medium, large, and jumbo. The seasons of the year were spring, summer, and fall. In the body of the table, the cells list the number of customers who fit both row and column titles. On the bottom and in the right margin are the totals.

	Spring	Summer	Fall	TOTALS
Small	24	22	18	64
Medium	23	28	19	70
Large	18	27	29	74
Jumbo	16	21	33	70
TOTALS	81	98	99	278

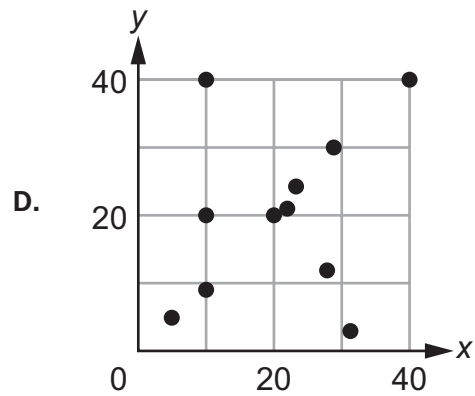
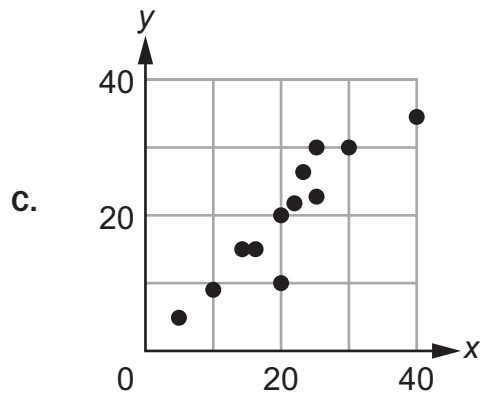
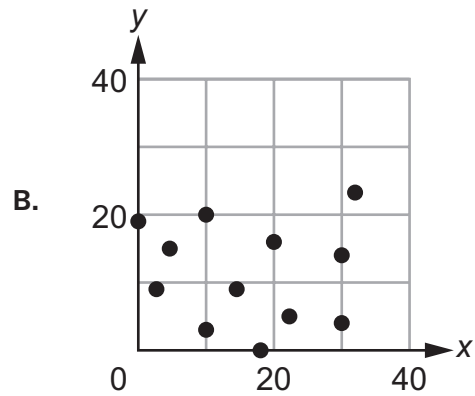
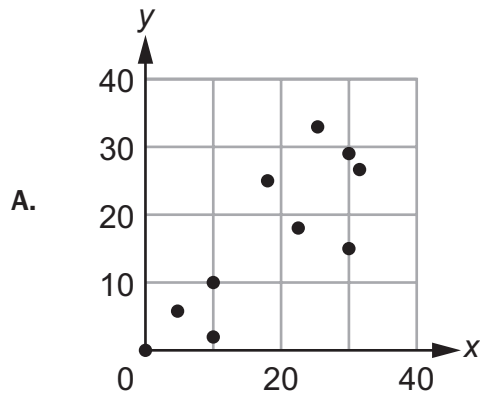
- In which season did the most customers purchase jumbo drinks?
- What percentage of those surveyed purchased small drinks?
- What percentage of those surveyed purchased medium drinks in the summer?
- What do you think the fast-food restaurant learned from its survey?

Solution:

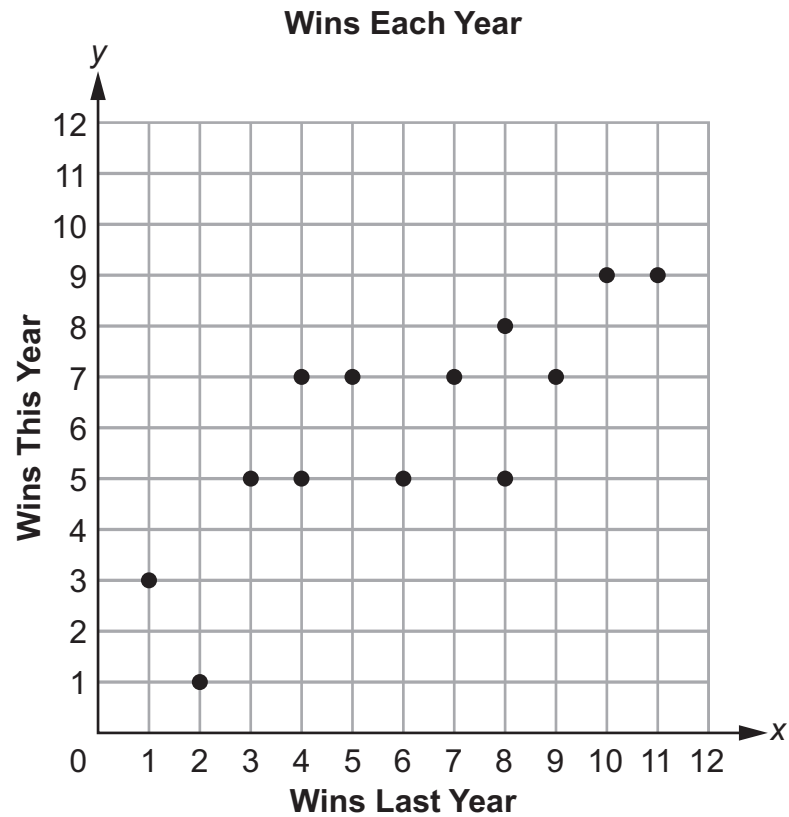
- The most customers purchase jumbo drinks in the fall.
- About twenty-three percent ($64/278 \approx 23\%$) of the 278 surveyed purchased small drinks.
- About ten percent ($28/278 \approx 10\%$) of those customers surveyed purchased medium drinks in the summer.
- The fast-food restaurant probably learned that customers tend to purchase the larger drinks in the fall and the smaller drinks in the spring and summer.

SAMPLE ITEMS

1. Which graph MOST clearly displays a set of data for which a linear function is the model of best fit?



2. This graph plots the number of wins last year and this year for a sample of professional football teams.



Which equation **BEST** represents a line that matches the trend of the data?

- A. $y = x + 2$
- B. $y = x + 7$
- C. $y = 0.6x - 0.2$
- D. $y = 0.6x + 2.4$

Answers to Unit 4.2 Sample Items

1. C 2. D

4.3 Interpret Linear Models

MGSE9-12.S.ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

MGSE9-12.S.ID.8 Compute (using technology) and interpret the correlation coefficient “ r ” of a linear fit. (For instance, by looking at a scatter plot, students should be able to tell if the correlation coefficient is positive or negative and give a reasonable estimate of the “ r ” value.) After calculating the line of best fit using technology, students should be able to describe how strong the goodness of fit of the regression is, using “ r .”

MGSE9-12.S.ID.9 Distinguish between correlation and causation.

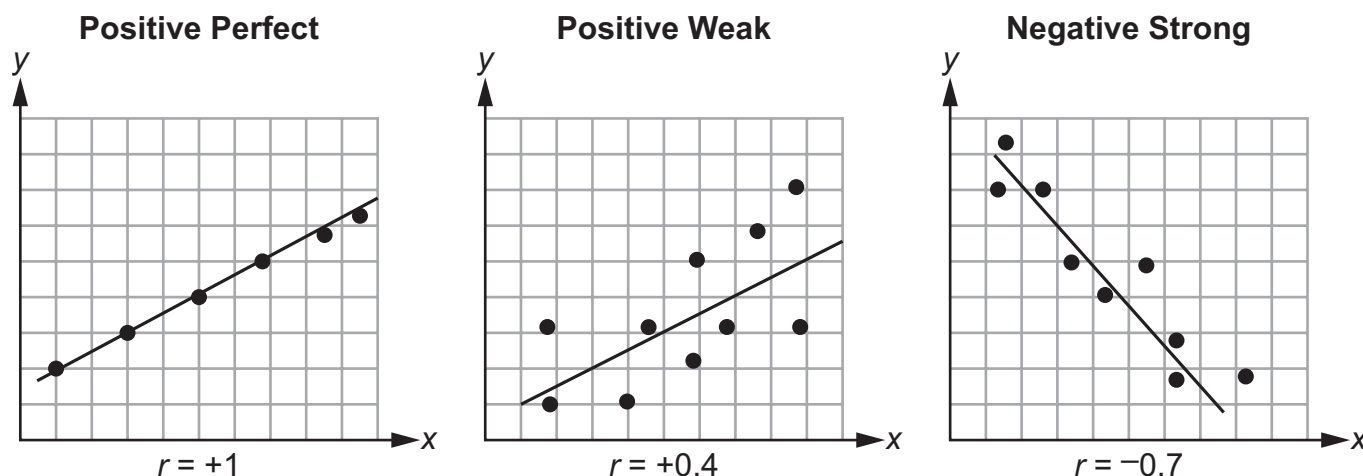
KEY IDEAS

Once a model for the scatter plot is determined, we can begin to analyze the correlation of the linear fit. We can also interpret the slope, or rate of change, and the constant term and distinguish between correlation and causation of the data.

A **correlation coefficient** is a measure of the strength of the linear relationship between two variables. It also indicates whether the dependent variable, y , grows along with x , gets smaller as x increases. The correlation coefficient is a number between -1 and $+1$ including -1 and $+1$. The letter r is usually used for the correlation coefficient. When the correlation is positive, the line of best fit will have a positive slope and both variables are growing. However, if the correlation coefficient is negative, the line of best fit has a negative slope and the dependent variable is decreasing. The numerical value is an indicator of how closely the data points are modeled by a linear function.

When using a calculator, use the same steps as you did to find the line of best fit. Notice there is a value, r , below the values for a and b . This is the correlation coefficient.

Examples:



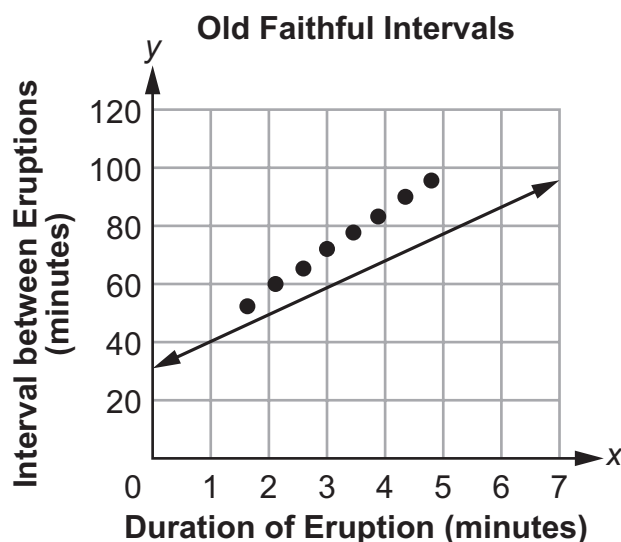
The correlation between two variables is related to the slope and the goodness of the fit of a regression line. However, data in scatter plots can have the same regression lines and very different correlations. The correlation's sign will be the same as the slope of the regression line. The correlation's value depends on the dispersion of the data points and their proximity to the line of best fit.

Example: Earlier we saw that the interval between eruptions of Old Faithful is related to the duration of the most recent eruption. Years ago, the National Park Service had a simple linear equation they used to help visitors determine when the next eruption would take place. Visitors were told to multiply the duration of the last eruption by 10 and add 30 minutes ($I = 10 \cdot D + 30$). We can look at a 2011 set of data for Old Faithful, with eight data points, and see how well the simple linear equation the National Park Service was using fits the 2011 data. The data points are from a histogram with intervals of 0.5 minute for x-values. The y-values are the average interval time for an eruption in that duration interval. The error distance is the difference between the interval and prediction.

Old Faithful Eruptions

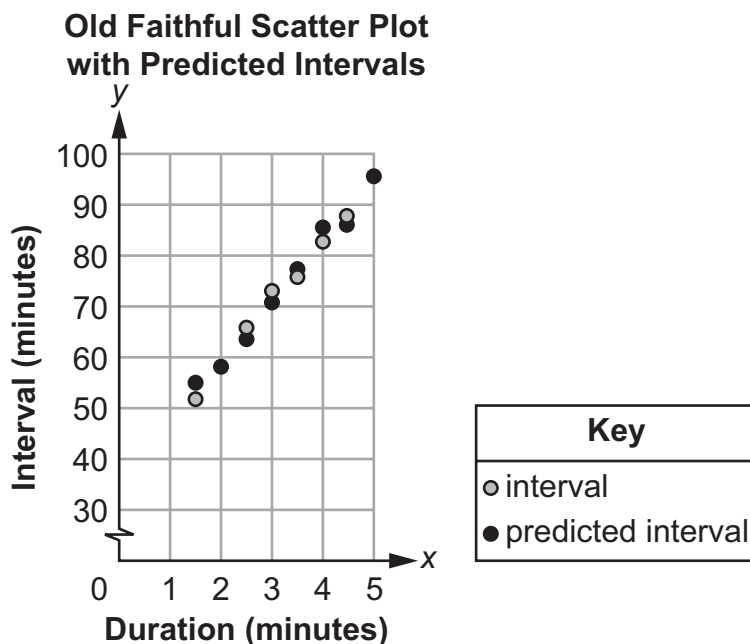
Duration (x)	Interval (y)	Prediction	Error Distances
1.50	51.00	45	6
2.00	58.00	50	8
2.50	65.00	55	10
3.00	71.00	60	11
3.50	76.00	65	11
4.00	82.00	70	12
4.50	89.00	75	14
5.00	95.00	80	15

The error distances display a clear pattern. They keep increasing by small increments. The Park Service's regression line on the scatter plot shows the same reality. The formula $I = 10 \cdot D + 30$ no longer works as a good predictor. In fact, it becomes a worse predictor as the length of eruptions increases. This is shown in the graph.



Instead of using the old formula, the National Park Service has a chart like the one in this example for visitors when they want to gauge how long it will be until the next eruption. We can take the chart the National Park Service uses and see what the new regression line would be. But first, does the Old Faithful Intervals scatter plot look like we should use a linear model? And, do the y-values of the data points in the chart have roughly a constant difference?

The answer to both questions is “yes.” The data points do look as though a linear model would fit. The differences in intervals are all 5s, 6s, and 7s. In cases like this, you can use technology to find a linear regression equation by entering the data points in the STAT feature of your calculator.



The technology determines data points that would fall on a new trend line that appear to fit the observed data points much better than the old line. The interval-predicting equation has new parameters for the model: $a = 12.36$ (up 2.36 minutes) and $b = 33.2$ (up 3.2 minutes). The new regression line would be $y = 12.36x + 33.2$. While the new regression line appears to come much closer to the observed data points, there are still error distances, especially for lesser duration times. The scientists at Yellowstone believe that there probably should be two regression lines now: one for use with shorter eruptions and another for longer eruptions. As we saw from the frequency distribution earlier, Old Faithful currently tends to have longer eruptions that are farther apart.

The technology also provides a correlation coefficient. From the picture of the regression points in Old Faithful Scatter Plot with Predicted Intervals, it looks like the number should be positive and fairly close to 1. Using the linear regression feature on the calculator, we get $r = 0.9992$. Indeed, the length of the interval between Old Faithful's eruptions is very strongly related to its most recent eruption duration. The direction is positive, confirming the longer the eruption, the longer the interval between eruptions.

It is very important to point out that the length of Old Faithful's eruptions does not directly cause the interval to be longer or shorter between eruptions. The reason it takes longer for Old Faithful to erupt again after a long eruption is not technically known. However, with a correlation coefficient so close to 1, the two variables are closely related to one another. However, you should never confuse correlation with causation. For example, research shows a correlation between income and age, but aging is not the reason for an increased income. Not all people earn more money the longer they live. Variables can be related to each other without one causing the other.

Correlation is when two or more things or events tend to occur at about the same time and might be associated with each other but are not necessarily connected by a cause/effect relationship. **Causation** is when one event occurs as a direct result of another event. For example, a runny nose and a sore throat may correlate to each other but that does not mean a sore throat causes a runny nose or a runny nose causes a sore throat. Another example is it is raining outside and the ground being wet. There is a correlation between how wet the ground gets and how much it rains. In this case, the rain is what caused the ground to get more wet, so there is causation.

Example: Consider the correlation between the age, in years, of a person and the income, in dollars, each person earns in the table below.

Age (years)	25	30	35	40	45	50
Income (dollars)	30,000	37,000	43,000	39,000	53,000	54,000

It appears there is a correlation between age and income and that a person's income increases as the person gets older. This does not mean age causes a person's income.

REVIEW EXAMPLES

- ◆ This scatter plot suggests a relationship between the variables age and income.

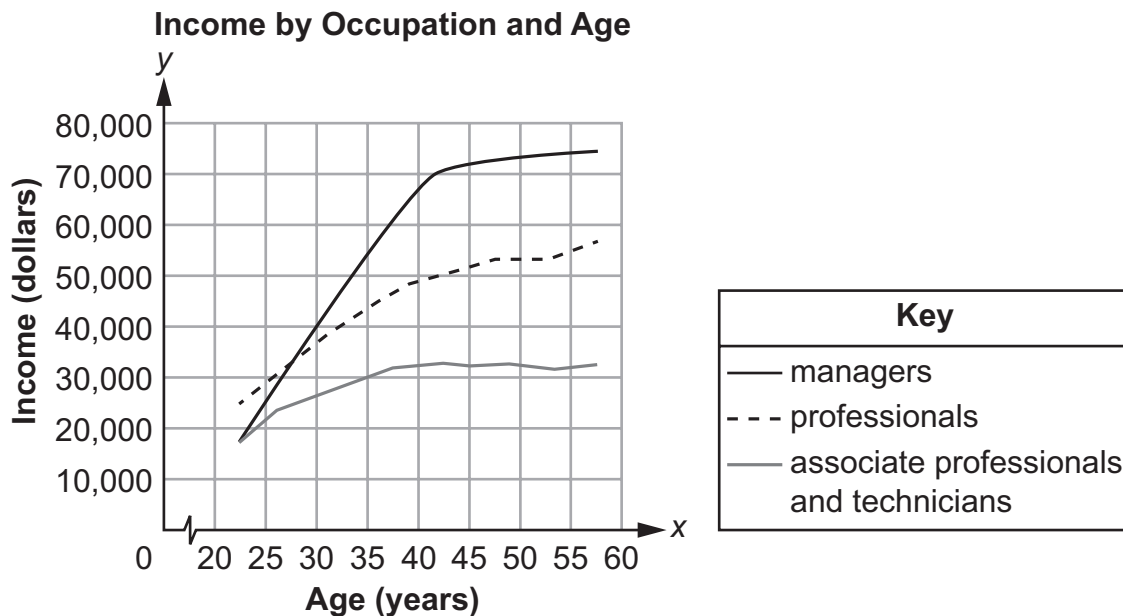


- What type of a relationship is suggested by the scatter plot (positive/negative, weak/strong)?
- What is the domain of ages considered by the researchers?
- What is the range of incomes?
- Do you think age causes income level to increase? Why or why not?

Solution:

- The scatter plot suggests a fairly strong positive relationship between age and yearly income.
- The domain of ages considered is 18 to 60 years.
- The range of incomes appears to be \$10,000 to \$70,000.
- No; the variables are related, but age does not cause income to increase.

- ◆ A group of researchers looked at income and age in Singapore. Their results are shown below. They used line graphs instead of scatter plots so they could consider the type of occupation of the wage earner.



- Does there appear to be a relationship between age and income?
- Do all three types of occupations appear to share the same benefit of aging when it comes to income?
- Does a linear model appear to fit the data for any of the occupation types?
- Does the relationship between age and income vary over a person's lifetime?

Solution:

- Yes, as people get older their income tends to increase.
- No. The incomes grow at different rates until age 40. For example, the managers' incomes grow faster than those of the other occupation types until age 40.
- No. The rate of growth appears to vary for all three occupations, making a linear model unsuitable for modeling this relationship over a longer domain.
- Yes, after about age 40, the income for each type of occupation grows slower than it did from age 22 to 40.

- ◆ An ice-cream shop uses a model to predict its daily ice-cream sales based on the daily high temperature. The table shows the daily high temperature, the daily sales, and the model's prediction for 8 days.

Ice-Cream Sales

Daily High Temperature (degrees F)	Daily Sales (dollars × 100)	Prediction (dollars × 100)	Error Distances
92	9	9.9	−0.9
93	12	10.2	1.8
88	9	8.6	0.4
96	10	11.2	−1.2
99	12	12.2	−0.2
92	11	9.9	1.1
86	7	7.9	−0.9
90	9	9.2	−0.2

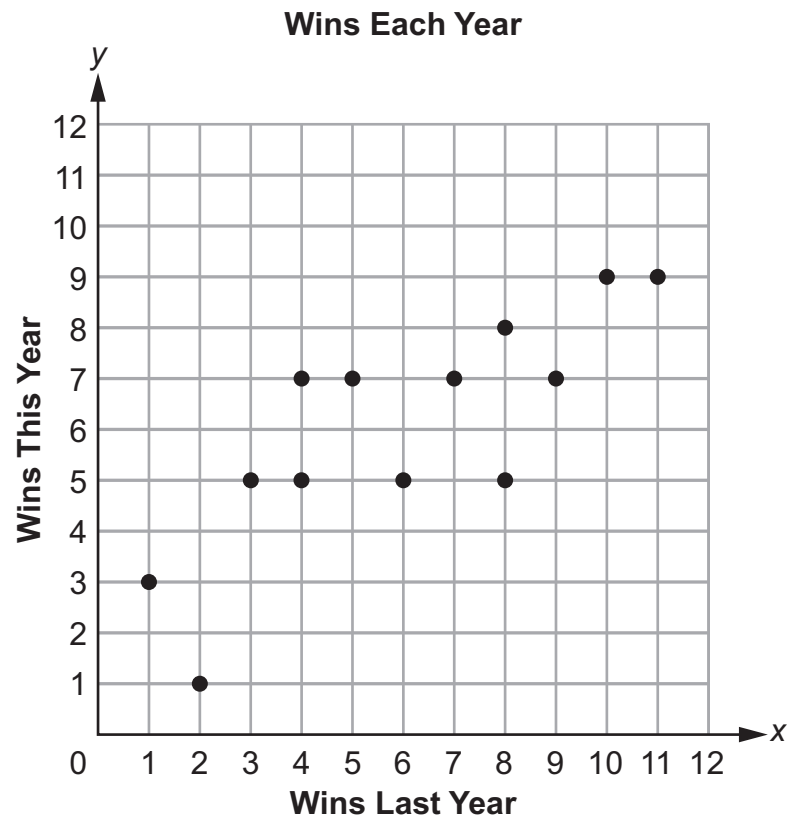
Explain why the ice-cream shop's model is or is not a good prediction for the daily ice-cream sales based on the daily high temperature.

Solution:

The prediction is a good model for the daily ice-cream sales based on the daily high temperature because all the values for the prediction are close to the actual daily sales with some of the values above the prediction and some of the values below the prediction.

SAMPLE ITEMS

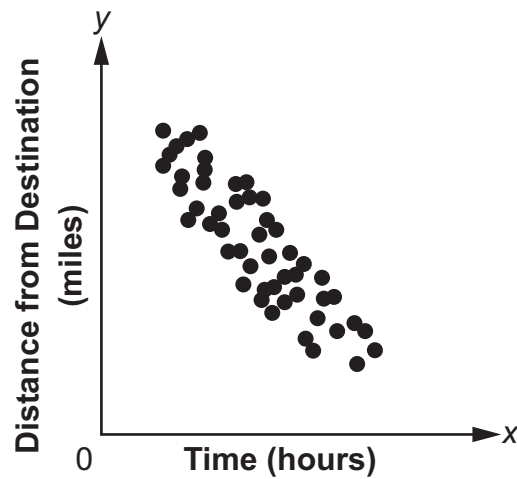
1. This graph plots the number of wins last year and this year for a sample of professional football teams.



Based on the line of best fit, which is the **BEST** prediction for wins this year for a team that won 4 games last year?

- A. 2
- B. 4
- C. 5
- D. 7

2. Which BEST describes the correlation of the two variables shown in the scatter plot?



- A. weak positive
- B. strong positive
- C. weak negative
- D. strong negative

3. Which statement describes an example of causation?

- A. When the weather becomes warmer, more meat is purchased at the supermarket.
- B. More people go to the mall when students go back to school.
- C. The greater the number of new television shows, the lesser the number of moviegoers.
- D. After operating costs are paid at a toy shop, as more toys are sold, more money is made.

4. To rent a carpet cleaner at the hardware store, there is a set fee and an hourly rate. The rental cost, c , can be determined using this equation when the carpet cleaner is rented for h hours.

$$c = 25 + 3h$$

Which of these is the hourly rate?

- A. 3
- B. $3h$
- C. 25
- D. $25h$

Answers to Unit 4.3 Sample Items

1. C 2. D 3. D 4. A

UNIT 5: TRANSFORMATIONS IN THE COORDINATE PLANE

In this unit, students review the definitions of three types of transformations that preserve distance and angle: rotations, reflections, and translations. They investigate how these transformations are applied in the coordinate plane as functions, mapping pre-image points (inputs) to image points (outputs). Using their knowledge of basic geometric figures and special polygons, they apply these transformations to obtain images of given figures. They also specify transformations that can be applied to obtain a given image from a given pre-image, including cases in which the image and pre-image are the same figure.

5.1 Experiment with Transformations in the Plane

MGSE9-12.G.CO.1 Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

MGSE9-12.G.CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

MGSE9-12.G.CO.3 Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

MGSE9-12.G.CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

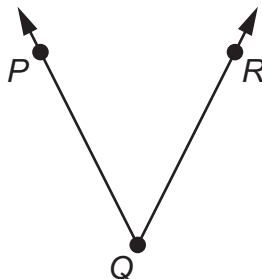
MGSE9-12.G.CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

KEY IDEAS

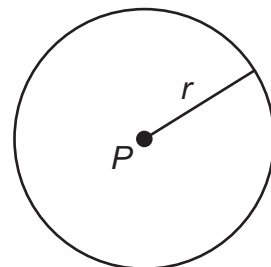
A **line segment** is part of a line; it consists of two points and all points between them. An **angle** is formed by two rays with a common endpoint. A **circle** is the set of all points in a plane that are equidistant from a given point, called the center; the fixed distance is the **radius**. **Parallel lines** are lines in the same plane that do not intersect. **Perpendicular lines** are two lines that intersect to form right angles.



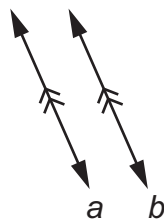
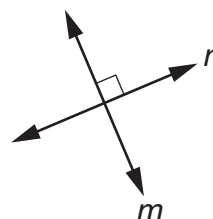
Line segment AB



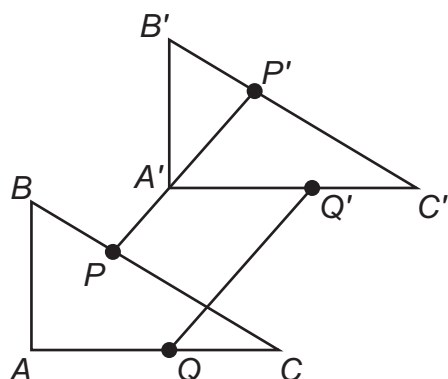
Angle PQR



Circle P with radius r

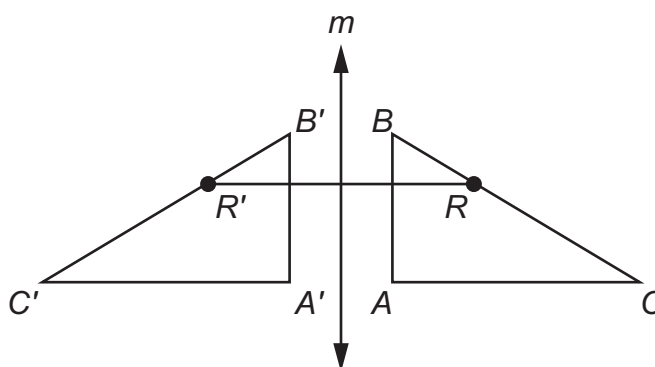

Parallel lines a and b

Perpendicular lines m and n

A **transformation** is an operation that maps, or moves, a pre-image onto an **image**. In each transformation defined below, it is assumed that all points and figures are in one plane. In each case, $\triangle ABC$ is the pre-image and $\triangle A'B'C'$ is the image.



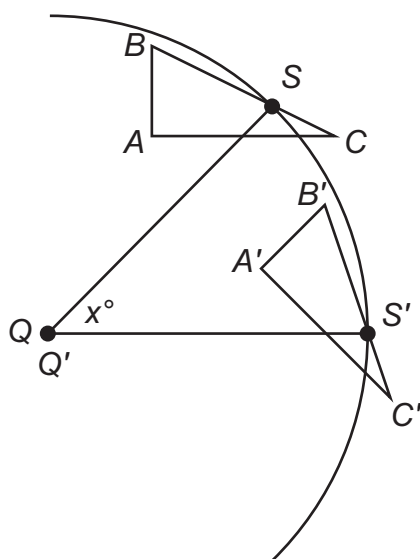
A **translation** maps every two points P and Q to points P' and Q' so that the following properties are true:

- $PP' = QQ'$
- $\overline{PP'} \parallel \overline{QQ'}$



A **reflection** across a line m maps every point R to R' so that the following properties are true:

- If R is not on m , then m is the perpendicular bisector of $\overline{RR'}$.
- If R is on m , then R and R' are the same point.

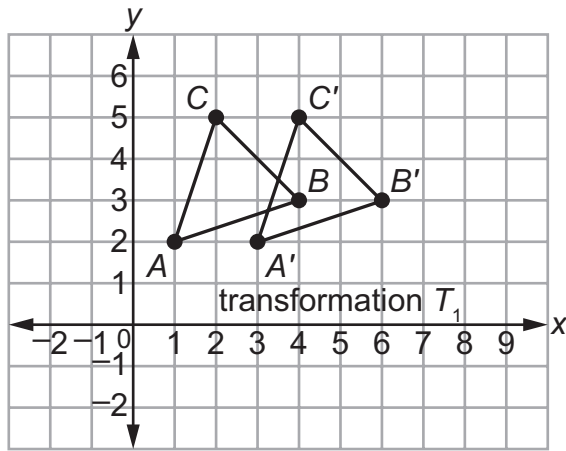


A **rotation** of x° about a point Q maps every point S to S' so that the following properties are true:

- $SQ = S'Q$ and $m\angle SQS' = x^\circ$.
- Pre-image point Q and image point Q' are the same.

Note: \overline{QS} and $\overline{QS'}$ are radii of a circle with center Q . x is called the **angle of rotation**.

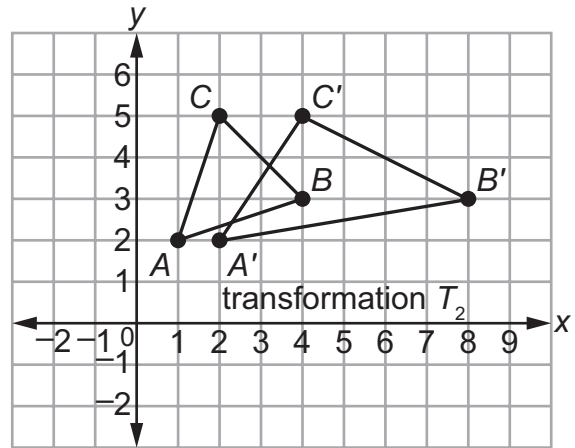
A transformation in a coordinate plane can be described as a function that maps pre-image points (inputs) to image points (outputs). Translations, reflections, and rotations all preserve distance and angle measure because, for each of those transformations, the pre-image and image are **congruent**. But some types of transformations do not preserve distance and angle measure, because the pre-image and image are not congruent.



$$T_1: (x, y) \rightarrow (x + 2, y)$$

T_1 translates $\triangle ABC$ to the right 2 units.

T_1 preserves distance and angle measure because $\triangle ABC \cong \triangle A'B'C'$.



$$T_2: (x, y) \rightarrow (2x, y)$$

T_2 stretches $\triangle ABC$ horizontally by the factor 2.

T_2 preserves neither distance nor angle measure.

If vertices are not named, then there might be more than one transformation that will accomplish a specified mapping. If vertices are named, then they must be mapped in a way that corresponds to the order in which they are named.

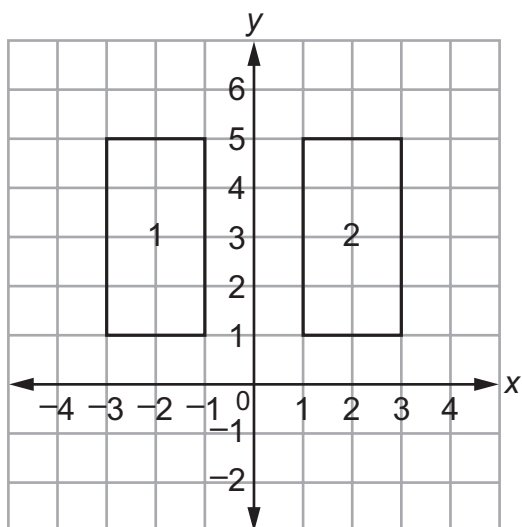
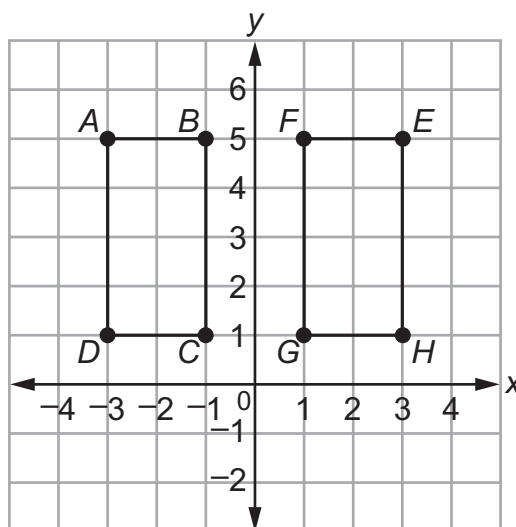


Figure 1 can be mapped to Figure 2 by either of these transformations:

- a reflection across the y -axis (the upper-left vertex in Figure 1 is mapped to the upper-right vertex in Figure 2)
- a translation 4 units to the right (the upper-left vertex in Figure 1 is mapped to the upper-left vertex in Figure 2)



$ABCD$ can be mapped to $EFGH$ by a reflection across the y -axis, but not by a translation.

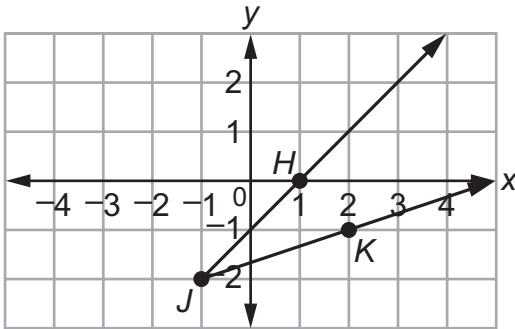
The mapping of $ABCD \rightarrow EFGH$ requires these vertex mappings:

$A \rightarrow E$, $B \rightarrow F$, $C \rightarrow G$, and $D \rightarrow H$.

REVIEW EXAMPLES

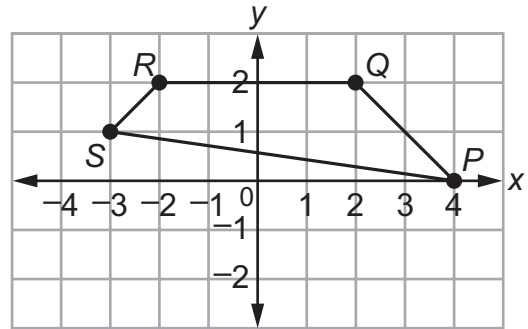
◆ Draw the image of each figure, using the given transformation.

a.



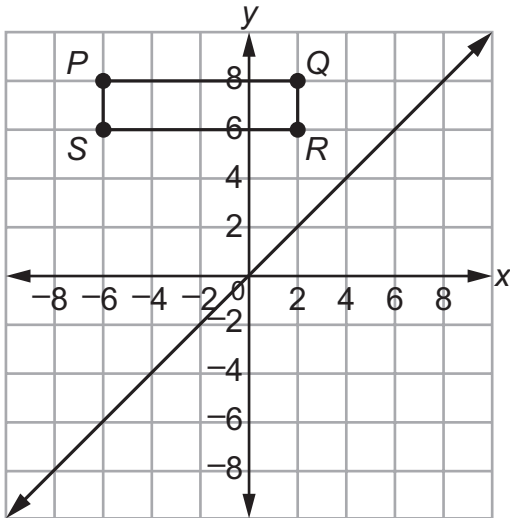
Use the translation $(x, y) \rightarrow (x - 3, y + 1)$.

b.



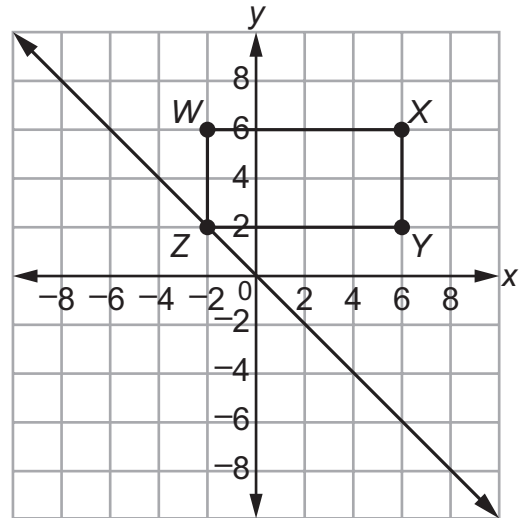
Reflect across the x -axis.

c.



Reflect across the line $y = x$.

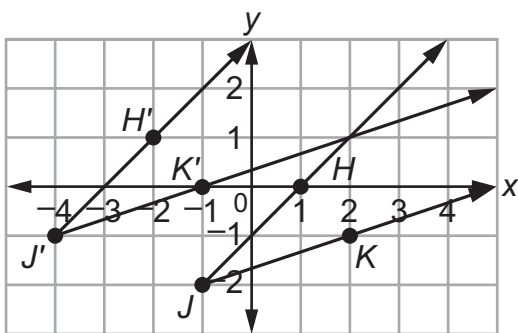
d.



Reflect across the line $y = -x$.

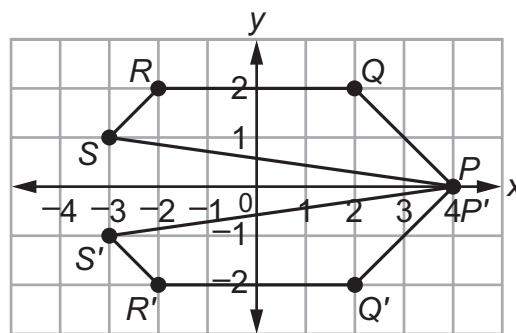
Solution:

a.



Identify the vertex and a point on each side of the angle. Translate each point 3 units left and 1 unit up. The image of given $\angle HJK$ is $\angle H'J'K'$.

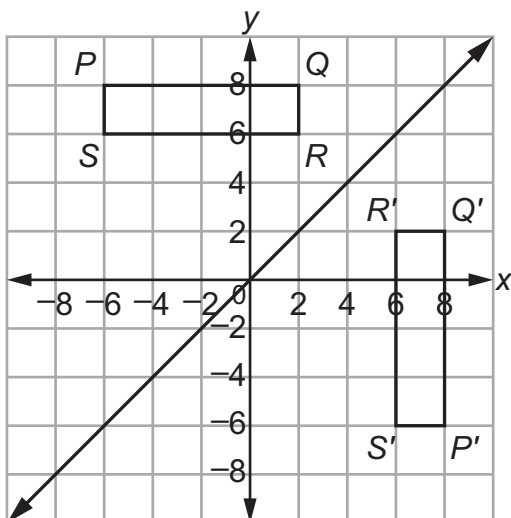
b.



Identify the vertices. The reflection image of each point (x, y) across the x -axis is $(x, -y)$.

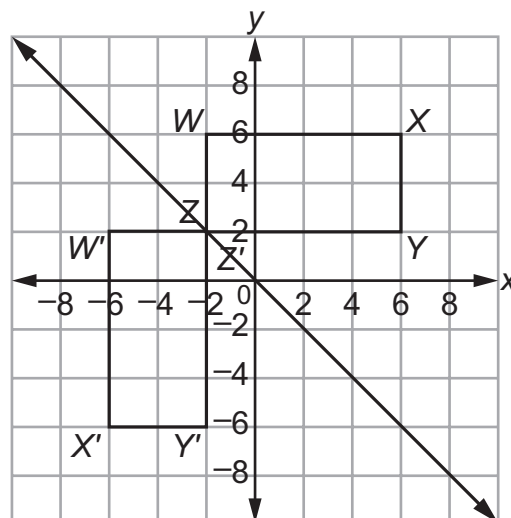
The image of given polygon $PQRS$ is $P'Q'R'S'$, where P and P' are the same.

c.



Identify the vertices. The reflection image of each point (x, y) across the line $y = x$ is (y, x) .

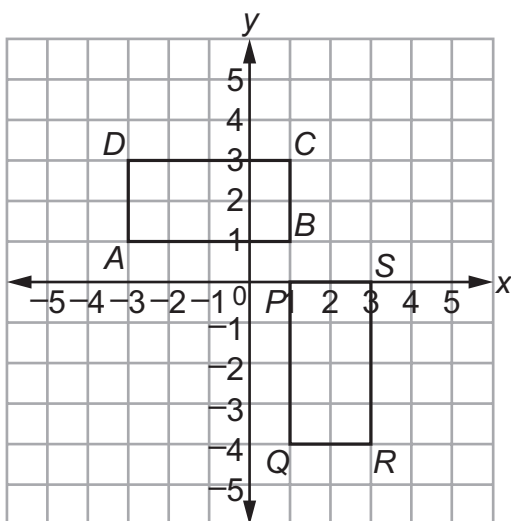
d.



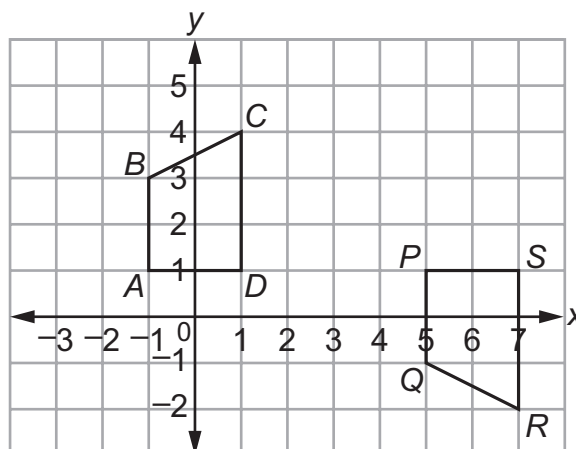
Identify the vertices. The reflection image of each point (x, y) across the line $y = -x$ is $(-y, -x)$.

- ◆ Specify a sequence of transformations that will map $ABCD$ to $PQRS$ in each case.

a.

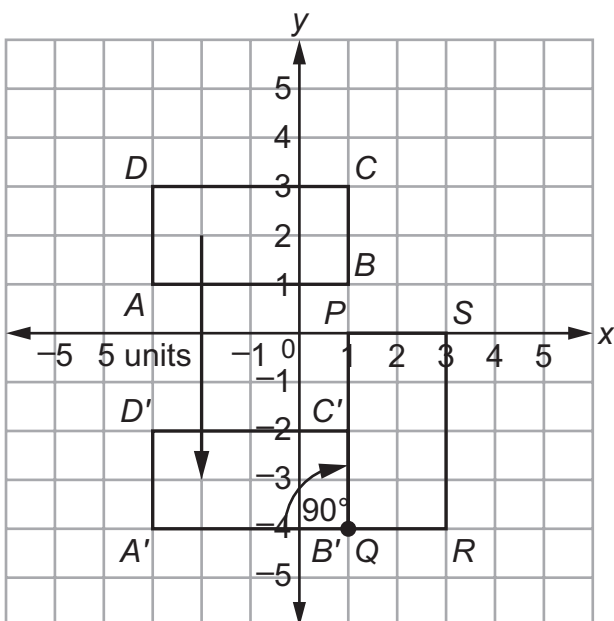


b.



Solution:

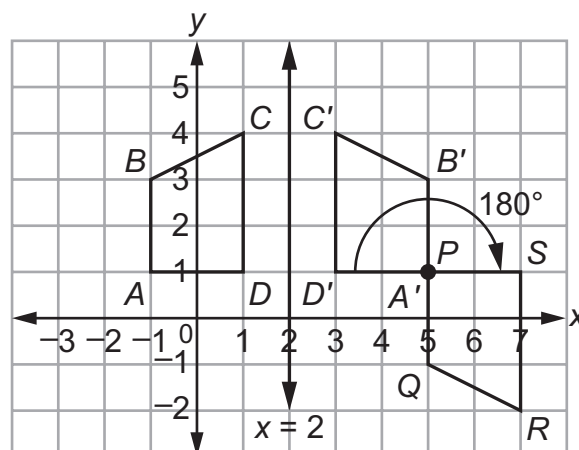
a.



Translate $ABCD$ down 5 units to obtain $A'B'C'D'$. Then rotate $A'B'C'D'$ clockwise 90° about point B' to obtain $PQRS$.

Note that B' and Q are the same point.

b.



Reflect $ABCD$ across the line $x = 2$ to obtain $A'B'C'D'$. Then rotate $A'B'C'D'$ 180° about point A' to obtain $PQRS$.

Note that A' and P are the same point.

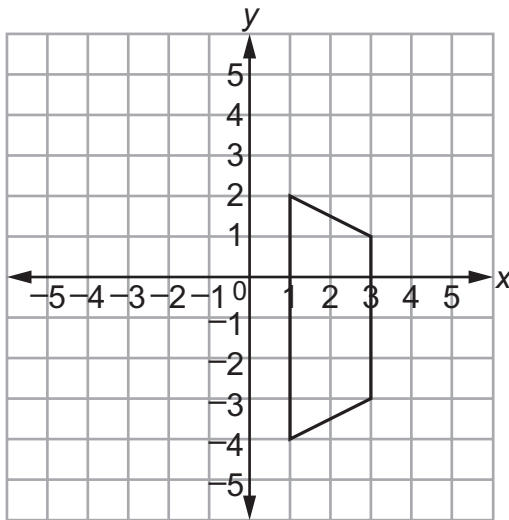
Note that there are other sequences of transformations that will also work for each case.

Important Tip

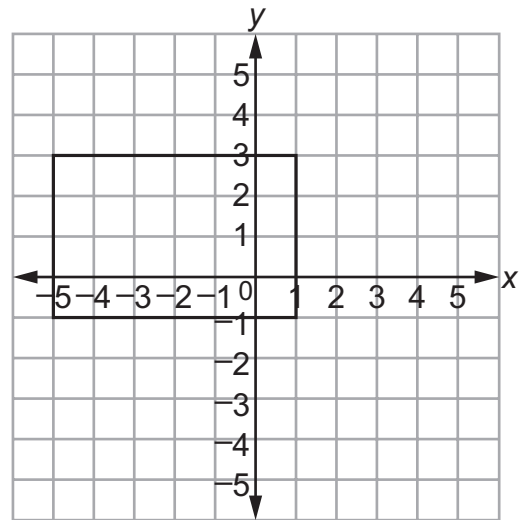
✍ A 180° rotation clockwise is equivalent to a 180° rotation counterclockwise.

◆ Describe every transformation that maps each given figure to itself.

a.

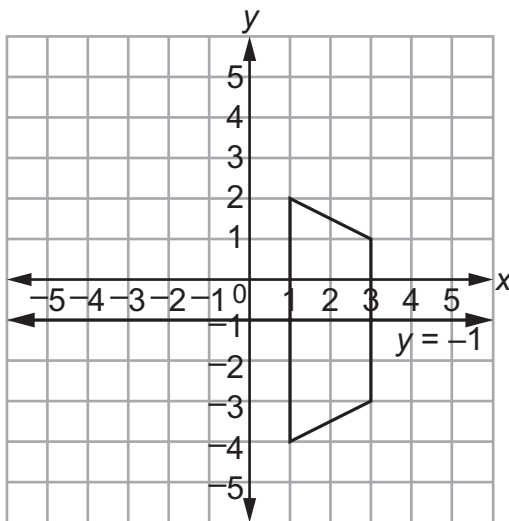


b.



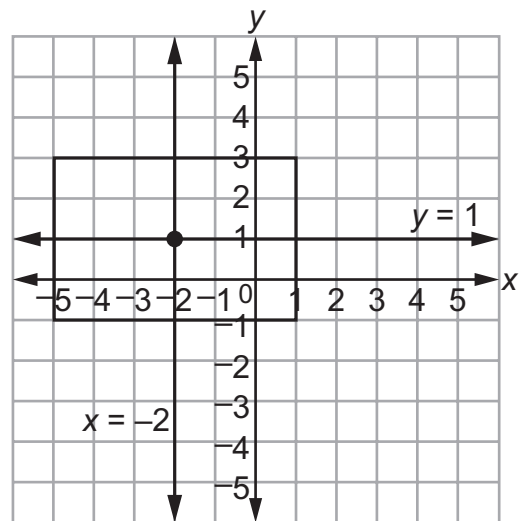
Solution:

a.



There is only one transformation:
Reflect the figure across the line $y = -1$.

b.



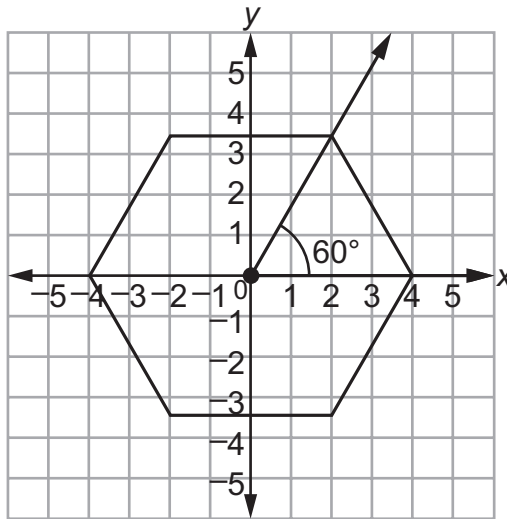
There are three transformations:

- Reflect across the line $y = 1$
- Reflect across the line $x = -2$
- Rotate 180° about the point $(-2, 1)$

- ◆ Describe every transformation that maps this figure to itself: a regular hexagon (6 sides) that is centered about the origin and that has a vertex at (4, 0).

Solution:

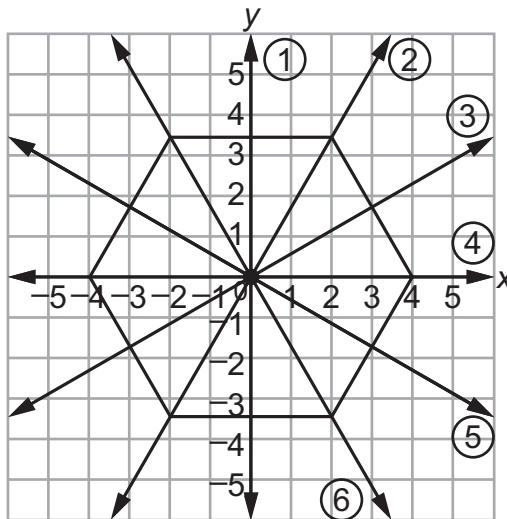
The angle formed by any two consecutive vertices and the center of the hexagon measures 60° because $\frac{360^\circ}{6} = 60^\circ$. So a rotation about the origin, clockwise or counterclockwise, of 60° , 120° , or any other multiple of 60° maps the hexagon to itself.



If a reflection across a line maps a figure to itself, then that line is called a **line of symmetry**.

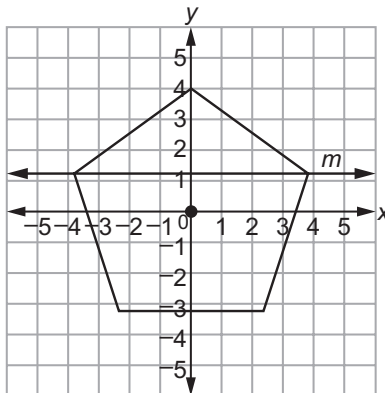
A regular hexagon has 6 lines of symmetry: 3 lines through opposite vertices and 3 lines through midpoints of opposite sides.

A reflection across any of the 6 lines of symmetry maps the hexagon to itself.



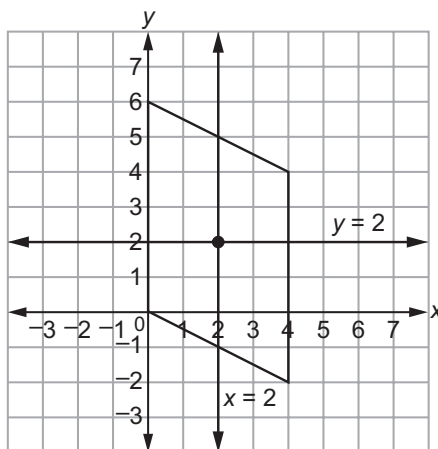
SAMPLE ITEMS

1. A regular pentagon is centered about the origin and has a vertex at $(0, 4)$.



Which transformation maps the pentagon to itself?

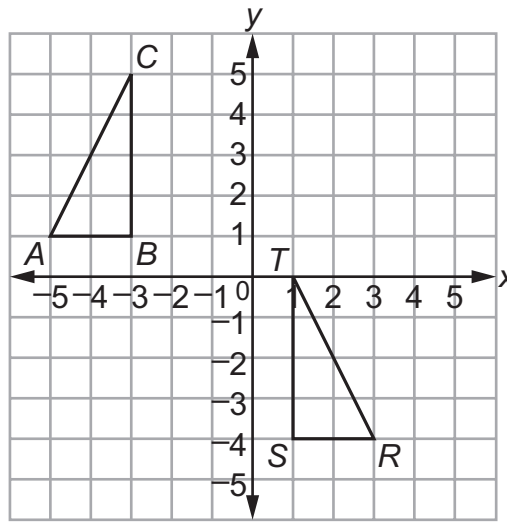
- A. a reflection across line m
 - B. a reflection across the x -axis
 - C. a clockwise rotation of 100° about the origin
 - D. a clockwise rotation of 144° about the origin
2. A parallelogram has vertices at $(0, 0)$, $(0, 6)$, $(4, 4)$, and $(4, -2)$.



Which transformation maps the parallelogram to itself?

- A. a reflection across the line $x = 2$
- B. a reflection across the line $y = 2$
- C. a rotation of 180° about the point $(2, 2)$
- D. a rotation of 180° about the point $(0, 0)$

3. Which sequence of transformations maps $\triangle ABC$ to $\triangle RST$?



- A. Reflect $\triangle ABC$ across the line $x = -1$. Then translate the result 1 unit down.
- B. Reflect $\triangle ABC$ across the line $x = -1$. Then translate the result 5 units down.
- C. Translate $\triangle ABC$ 6 units to the right. Then rotate the result 90° clockwise about the point (1, 1).
- D. Translate $\triangle ABC$ 6 units to the right. Then rotate the result 90° counterclockwise about the point (1, 1).

Answers to Unit 5.1 Sample Items

1. D 2. C 3. B

UNIT 6: CONNECTING ALGEBRA AND GEOMETRY THROUGH COORDINATES

The focus of this unit is to have students analyze and prove geometric properties by applying algebraic concepts and skills on a coordinate plane. Students learn how to prove the fundamental theorems involving parallel and perpendicular lines and their slopes, applying both geometric and algebraic properties in these proofs. They also learn how to prove other theorems, applying them to figures with specified numerical coordinates. (A theorem is any statement that is proved or can be proved. Theorems can be contrasted with postulates, which are statements that are accepted without proof.)

6.1 Use Coordinates to Prove Simple Geometric Theorems Algebraically

MGSE9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle.

MGSE9-12.G.GPE.5 Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).

MGSE9-12.G.GPE.6 Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

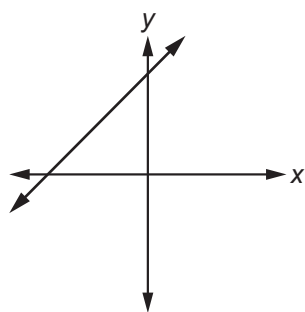
MGSE9-12.G.GPE.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

KEY IDEAS

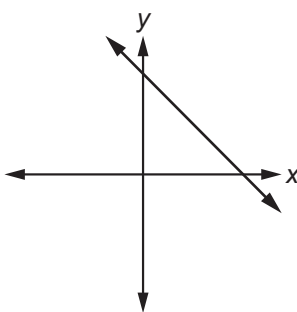
The **distance** between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The **slope** of the line through points (x_1, y_1) and (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.

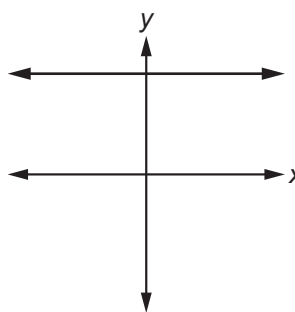
Slopes can be positive, negative, 0, or undefined.



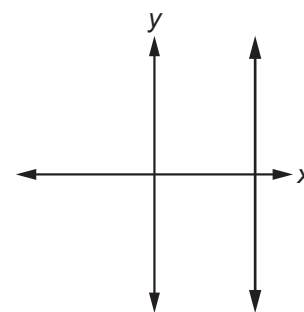
A line with a **positive slope** slants up to the right.



A line with a **negative slope** slants down to the right.



A line with a **slope of 0** is horizontal.



A line with an **undefined slope** is vertical.

Lines and their slopes are related by the following properties:

- Two nonvertical lines are **parallel** if and only if they have equal slopes but different y-intercepts. Parallel lines are denoted by the symbol \parallel . For example, line m is parallel to line n can be written as $m \parallel n$.

- b. Two nonvertical lines are **perpendicular** if and only if the product of their slopes is -1 . Perpendicular lines are denoted by the symbol \perp . For example, line m is perpendicular to line n can be written as $m \perp n$.

Some useful **properties of proportions** state that all of the following are equivalent:

$$\frac{a}{b} = \frac{c}{d} \quad ad = bc \quad \frac{a}{c} = \frac{b}{d}$$

Example: Use the multiplication property of equality to multiply each side of the proportion. Because

$\frac{5}{3} = \frac{10}{6}$ you can also write $5 \cdot 6 = 3 \cdot 10$ and $\frac{5}{10} = \frac{3}{6}$. Multiply each side by 18, a common multiple of 3 and 6.

$$\frac{5}{3}(18) = (18)\frac{10}{6}$$

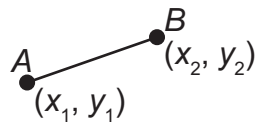
$$5 \cdot 6 = 3 \cdot 10$$

Multiply each side by $\frac{3}{10}$.

$$\frac{3}{10} \cdot \frac{5}{3} = \frac{3}{10} \cdot \frac{10}{6}$$

$$\frac{5}{10} = \frac{3}{6}$$

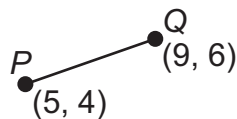
A **directed line segment** is a line segment from one point to another point in the coordinate plane.



Example: Notice on \overline{PQ} we can subtract to find the difference of the x- and y-values of Q and P:

$(9 - 5, 6 - 4) = (4, 2)$. They tell you that a “route” from P to Q is 4 units right and 2 units up. Note that

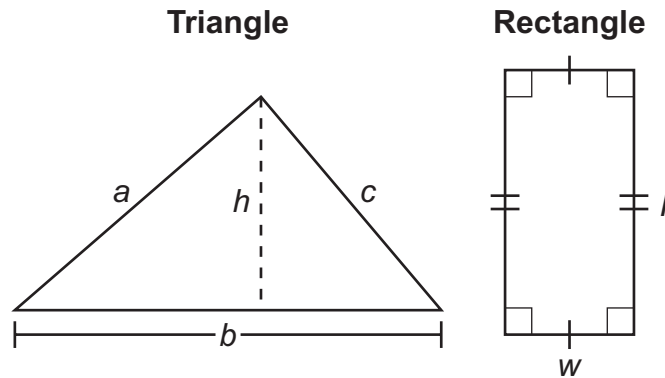
these are used in the slope: $\frac{6 - 4}{9 - 5} = \frac{2}{4} = \frac{1}{2}$.



Important Tip

✍ When using directed line segments, pay close attention to the beginning and end points of the line. For example, the directed line segments \overline{PQ} and \overline{QP} have the same length, but different directions.

The **perimeter** of a polygon is the sum of the lengths of the sides. The **area** of a polygon is the number of square units enclosed by the polygon.



For a triangle with side lengths a , b , c , with side b as the base and height h , the perimeter P and area A are

$$P = a + b + c$$


$$A = \frac{1}{2}bh$$

For a rectangle with length l and width w , the perimeter P and area A are

$$P = 2l + 2w$$

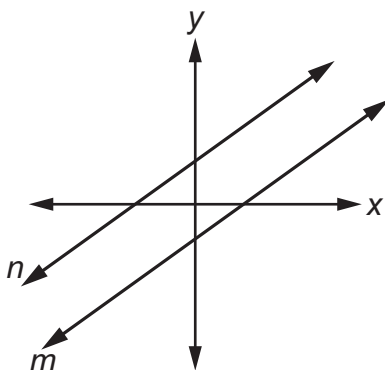
$$A = lw$$

Important Tip

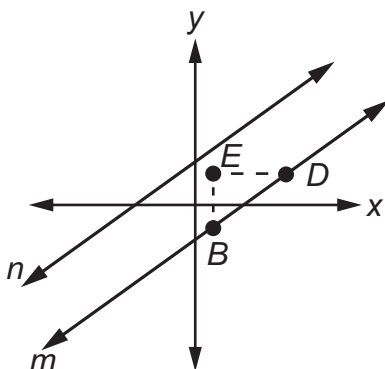
 In a triangle, any side can be used as the base. The corresponding height is the altitude drawn to the line containing that base. In a right triangle, the legs can be used as the base and height.

REVIEW EXAMPLES

- ◆ Follow the steps below to prove that if two nonvertical lines are parallel, then they have equal slopes.
- a. Let the straight lines n and m be parallel. Sketch these on a coordinate grid.

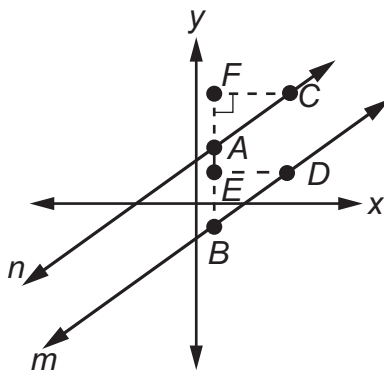


- b. Plot any points B and D on line m and the point E so that segment BE is the rise and segment ED is the run of the slope of line m . (A straight line can have only one slope.)



That is, slope of line m is $\frac{BE}{ED}$.

- c. Draw the straight line BE so that it intersects line n at point A and extends to include point F such that segment FC is perpendicular to BE .



- d. What is the slope of line n ?

Solution:

The slope is $\frac{AF}{FC}$.

- e. Line BF is the _____ of lines m and n , so $\angle EBD$ and $\angle FAC$ are _____ angles, so $\angle EBD$ _____ $\angle FAC$.

Solution:

transversal, corresponding, congruent

- f. Why is it true that $\angle DEB \cong \angle AFC$?

Solution:

The angles are right angles.

- g. Now, $\triangle DEB$ and $\triangle CFA$ are similar, so the ratio of their sides is proportional. Write the proportion that relates the vertical leg to the horizontal leg of the triangles.

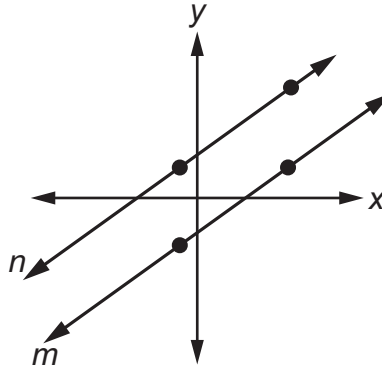
Solution:

$$\frac{BE}{ED} \cong \frac{AF}{FC}$$

- h. Note that this proportion shows the slope of line m is the same as the slope of line n . Therefore, parallel lines have the same slope.

Prove that if two nonvertical lines have equal slopes, then they are parallel.

Solution:



Use a proof by contradiction. Assume that the lines have equal slopes but are not parallel—that is, assume the lines intersect. If you can show this is not true, it is equivalent to proving the original statement.

Write the equations for both lines. The slopes are the same, so use m for the slope of each line. The two lines are different, so $b_1 \neq b_2$.

Equation for line n : $y = mx + b_1$

Equation for line m : $y = mx + b_2$

Solve a system of equations to find the point of intersection. Both equations are solved for y , so use substitution.

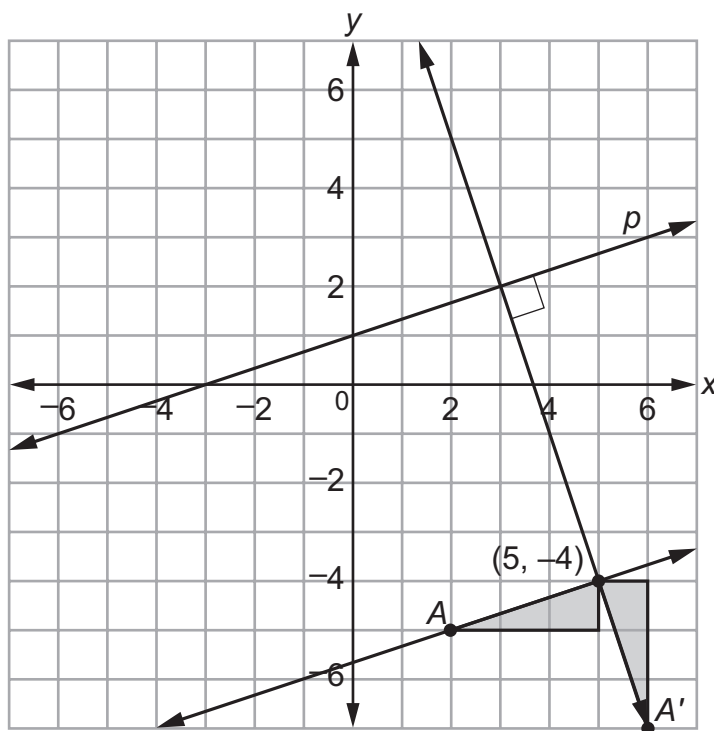
$$\begin{aligned} mx + b_1 &= mx + b_2 \\ b_1 &= b_2 \end{aligned}$$

Because it was assumed that $b_1 \neq b_2$, this is a contradiction. So the original statement is true—if two nonvertical lines have equal slopes, then they are parallel.

- ◆ The line p is represented by the equation $y = \frac{1}{3}x + 1$. What is the equation of the line that is perpendicular to line p and passes through the point $(5, -4)$?

Solution:

On a coordinate grid, graph the line $y = \frac{1}{3}x + 1$ and the line parallel to $y = \frac{1}{3}x + 1$ that passes through the point $(5, -4)$, which is $y = \frac{1}{3}x - \frac{17}{3}$. Next, choose any point on the parallel line, $(2, -5)$ for example, and name the point A . Form a right triangle with the point $(5, -4)$ and point A . Rotate the triangle 90° around the point $(5, -4)$. Use the hypotenuse of the new triangle to draw the perpendicular line. The new line has a slope of -3 and passes through the point $(5, -4)$.



The slope-intercept form of the equation of a line is $y = mx + b$. Substitute -3 for m . The line that is perpendicular to line p passes through the point $(5, -4)$. So substitute 5 in for x and -4 for y . Solve for b .

$$-4 = -3(5) + b$$

$$-4 = -15 + b$$

$$11 = b$$

The equation of the line that is perpendicular to line p and passes through the point $(5, -4)$ is $y = -3x + 11$.

- ◆ For what value of n are the lines $7x + 3y = 8$ and $nx + 3y = 8$ perpendicular?

Solution:

The two lines will be perpendicular when the slopes are opposite reciprocals.

First, find the slope of the line $7x + 3y = 8$.

$$7x + 3y = 8$$

$$3y = -7x + 8$$

$$y = -\frac{7}{3}x + \frac{8}{3}$$

The slope is $-\frac{7}{3}$.

Next, find the slope of the line $nx + 3y = 8$, in terms of n .

$$nx + 3y = 8$$

$$3y = -nx + 8$$

$$y = -\frac{n}{3}x + \frac{8}{3}$$

The slope is $-\frac{n}{3}$.

The opposite reciprocal of $-\frac{7}{3}$ is $\frac{3}{7}$. Find the value of n that makes the slope of the second line $\frac{3}{7}$.

$$-\frac{n}{3} = \frac{3}{7}$$

$$-n = \frac{9}{7}$$

$$n = -\frac{9}{7}$$

When $n = -\frac{9}{7}$, the two lines are perpendicular.

- ◆ Quadrilateral $ABCD$ has vertices $A(4, 0)$, $B(3, 3)$, $C(-3, 1)$, and $D(-2, -2)$. Prove that $ABCD$ is a rectangle.

Solution:

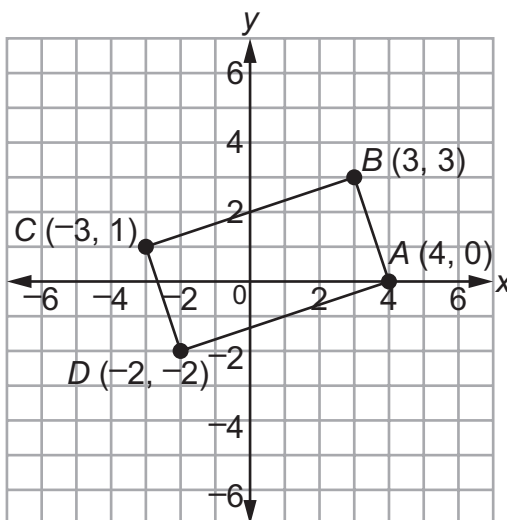
The slopes of the sides are

$$\overline{AB}: \frac{3 - 0}{3 - 4} = \frac{3}{-1} = -3$$

$$\overline{BC}: \frac{1 - 3}{-3 - 3} = \frac{-2}{-6} = \frac{1}{3}$$

$$\overline{CD}: \frac{-2 - 1}{-2 + 3} = \frac{-3}{1} = -3$$

$$\overline{DA}: \frac{0 + 2}{4 + 2} = \frac{2}{6} = \frac{1}{3}$$



$\overline{AB} \parallel \overline{CD}$ because they have equal slopes.

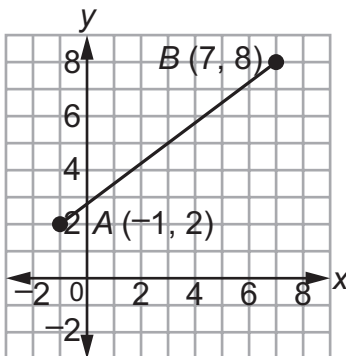
$\overline{BC} \parallel \overline{DA}$ because they have equal slopes.

So $ABCD$ is a parallelogram because both pairs of opposite sides are parallel.

$\overline{AB} \perp \overline{BC}$ because the product of their slopes is -1 : $-3 \cdot \frac{1}{3} = -1$.

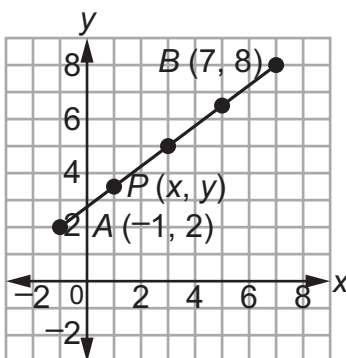
Therefore, $ABCD$ is a rectangle because it is a parallelogram with a right angle.

- ◆ Given the points $A(-1, 2)$ and $B(7, 8)$, find the coordinates of point P on directed line segment \overline{AB} that partitions \overline{AB} in the ratio 1:3.



Solution:

Point P partitions \overline{AB} in the ratio 1:3 if P is on \overline{AB} . This means that you need to divide \overline{AB} into 4 equal parts, since $\overline{AP} = \text{one-fourth of } \overline{AB}$.



Let $P(x, y)$ be on \overline{AB} . Solve two equations to find x and y , where (x_1, y_1) is the starting point, (x_2, y_2) is the ending point, and $\frac{a}{(a+b)} = \frac{1}{4}$.

$$(x, y) = \left(x_1 + \frac{a}{a+b}(x_2 - x_1), y_1 + \frac{a}{a+b}(y_2 - y_1) \right)$$

$$(x, y) = \left(-1 + \frac{1}{4}(7 - (-1)), 2 + \frac{1}{4}(8 - 2) \right)$$

$$x = -1 + \frac{1}{4}(7 - (-1)) \qquad y = 2 + \frac{1}{4}(8 - 2)$$

$$x = -1 + \frac{1}{4}(8) \qquad y = 2 + \frac{1}{4}(6)$$

$$x = -1 + 2 \qquad y = 2 + \frac{3}{2}$$

$$x = 1$$


$$y = \frac{7}{2}$$

The coordinates of P are $\left(1, \frac{7}{2}\right)$.

Here is another method for partitioning segment AB in the ratio 1:3 from $A(x_1, y_1)$ to $B(x_2, y_2)$.

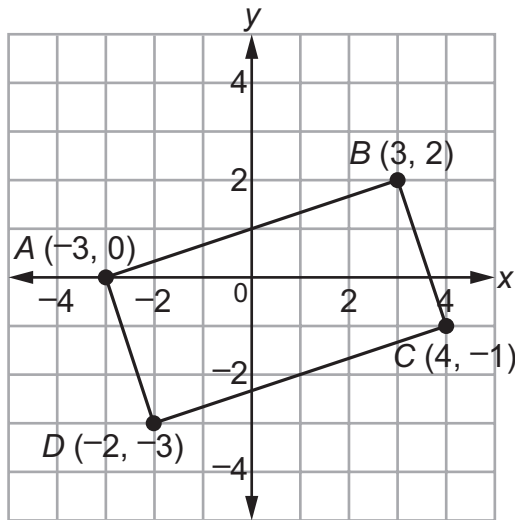
$$\begin{aligned} (x, y) &= \frac{bx_1 + ax_2}{b+a}, \frac{by_1 + ay_2}{b+a} \\ &= \frac{3(-1) + 1(7)}{3+1}, \frac{3(2) + 1(8)}{3+1} \\ &= \frac{-3+7}{4}, \frac{6+8}{4} \\ &= \frac{4}{4}, \frac{14}{4} \\ &= \left(1, \frac{7}{2}\right) \end{aligned}$$

Important Tip

 Be careful when using directed line segments. If point P partitions \overline{AB} in the ratio 1:3, then point P partitions \overline{BA} in the ratio 3:1.

- ◆ Find the area of rectangle $ABCD$ with vertices $A(-3, 0)$, $B(3, 2)$, $C(4, -1)$, and $D(-2, -3)$.

Solution:



One strategy is to use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the length and width of the rectangle.

$$AB = \sqrt{(3 - (-3))^2 + (2 - 0)^2} = \sqrt{36 + 4} = \sqrt{40}$$

$$BC = \sqrt{(4 - 3)^2 + (-1 - 2)^2} = \sqrt{1 + 9} = \sqrt{10}$$

The length of the rectangle is usually considered to be the longer side. Therefore, the length of the rectangle is $\sqrt{40}$ and the width is $\sqrt{10}$.

NOTE: Other strategies are possible to find the length of AB and BC such as using the Pythagorean theorem.

Use the area formula.

$$A = lw$$

$$A = (\sqrt{40})(\sqrt{10})$$

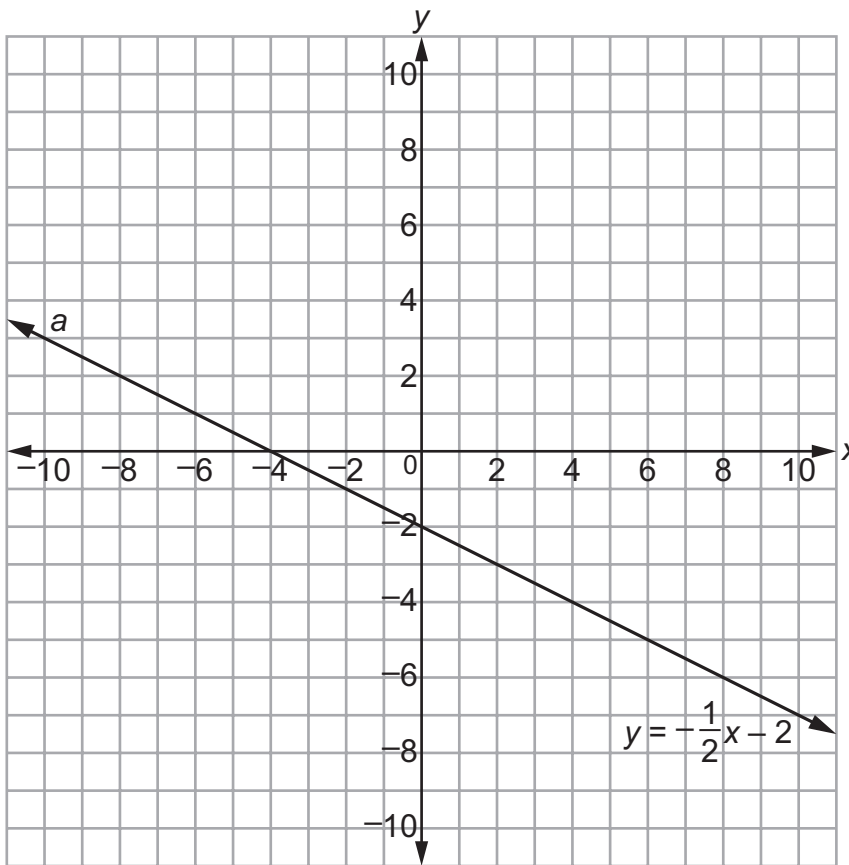
$$A = \sqrt{400}$$

$$A = 20$$

The area of the rectangle is 20 square units.

SAMPLE ITEMS

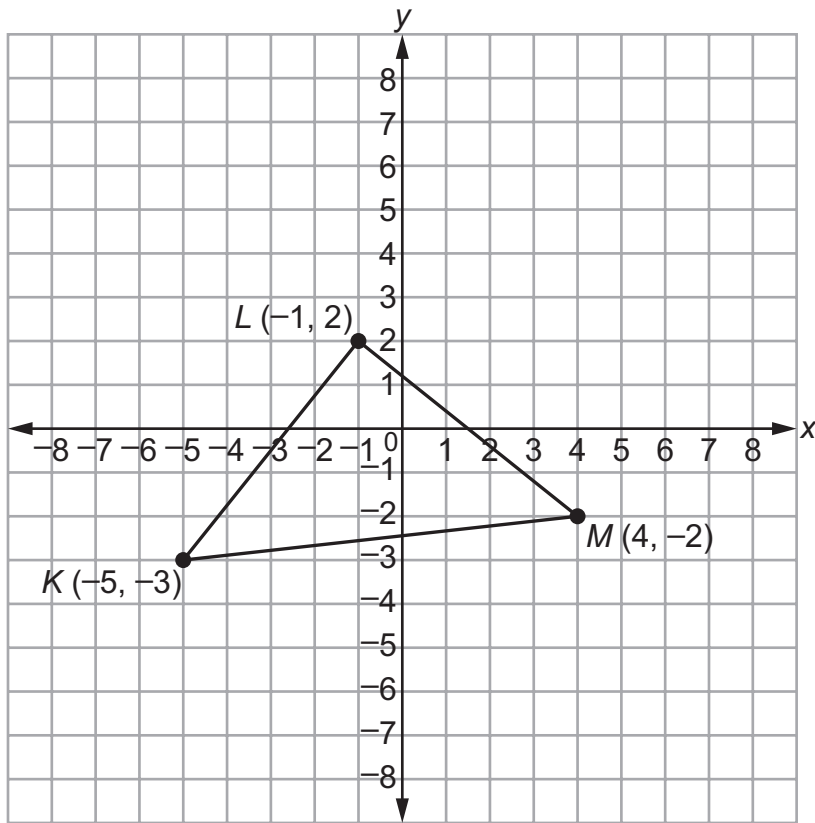
1. An equation of line a is $y = -\frac{1}{2}x - 2$.



Which is an equation of the line that is perpendicular to line a and passes through the point $(-4, 0)$?

- A. $y = -\frac{1}{2}x + 2$
- B. $y = -\frac{1}{2}x + 8$
- C. $y = 2x - 2$
- D. $y = 2x + 8$

2. Triangle KLM has vertices as shown.



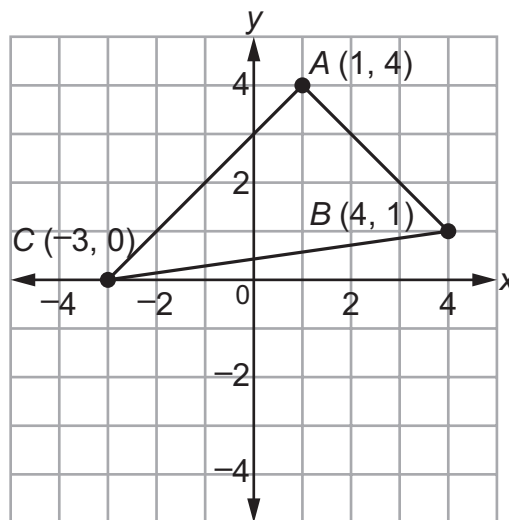
Which relationship between the slopes of triangle KLM proves that it is a right triangle?

- A. $\overline{LK} \perp \overline{KM}$ because the product of the slopes of these segments is -1 .
- B. $\overline{KL} \perp \overline{LM}$ because the product of the slopes of these segments is -1 .
- C. $\overline{KL} + \overline{LM}$ because the sum of the slopes of these segments is less than 1 .
- D. $\overline{LK} + \overline{KM}$ because the sum of the slopes of these segments is greater than 1 .

3. Given points $P(11, 2)$ and $Q(-13, -5)$, what are the coordinates of the point on directed line segment \overline{PQ} that partitions \overline{PQ} in the ratio 3:2?

- A. $\left(-\frac{17}{5}, -\frac{11}{5}\right)$
B. $\left(\frac{7}{24}, -\frac{29}{24}\right)$
C. $\left(-1, -\frac{3}{2}\right)$
D. $\left(-5, -\frac{8}{3}\right)$

4. Triangle ABC has vertices as shown.



What is the area of the triangle?

- A. $\sqrt{72}$ square units
B. 12 square units
C. $\sqrt{288}$ square units
D. 24 square units

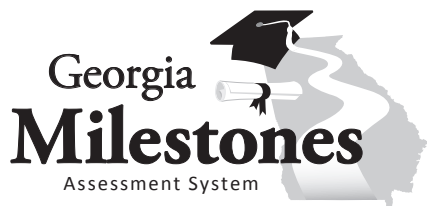
Answers to Unit 6.1 Sample Items

1. D 2. B 3. A 4. B

COORDINATE ALGEBRA ADDITIONAL PRACTICE ITEMS

This section has two parts. The first part is a set of 27 sample items for Coordinate Algebra. The second part contains a table that shows for each item the standard assessed, the DOK level, the correct answer (key), and a rationale/explanation about the key and distractors. The sample items can be utilized as a mini-test to familiarize students with the item formats found on the assessment.

All example and sample items contained in this guide are the property of the Georgia Department of Education.



Coordinate Algebra Formula Sheet

Below are the formulas you may find useful as you take the test. However, you may find that you do not need to use all of the formulas. You may refer to this formula sheet as often as needed.

Linear Formulas

Slope Formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Linear Equations

Slope-intercept Form: $y = mx + b$

Point-slope Form: $y - y_1 = m(x - x_1)$

Standard Form: $Ax + By = C$

Arithmetic Sequence Formulas

Recursive: $a_n = a_{n-1} + d$

Explicit: $a_n = a_1 + d(n - 1)$

Geometry Formulas

Perimeter

The perimeter of a polygon is equal to the sum of the lengths of its sides.

Area

Triangle $A = \frac{1}{2}bh$

Rectangle $A = bh$

Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Coordinates of point which partitions a directed line segment AB at the ratio of $a:b$ from $A(x_1, y_1)$ to $B(x_2, y_2)$

$$(x, y) = \left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a} \right)$$

OR

$$(x, y) = \left(x_1 + \frac{a}{a + b}(x_2 - x_1), y_1 + \frac{a}{a + b}(y_2 - y_1) \right)$$

Average Rate of Change

The change in the y -value divided by the change in the x -value for two distinct points on a graph.

Exponential Formulas

Exponential Equation

$$y = ab^x$$

Geometric Sequence Formulas

Recursive: $a_n = r(a_{n-1})$

Explicit: $a_n = a_1 \cdot r^{n-1}$

Compound Interest Formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Statistics Formulas

Mean

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n}$$

Interquartile Range

$$IQR = Q_3 - Q_1$$

The difference between the first quartile and third quartile of a set of data.

Mean Absolute Deviation

$$\frac{\sum_{i=1}^n |x_i - \bar{x}|}{n}$$

The sum of the distances between each data value and the mean, divided by the number of data values.

You can find mathematics formula sheets on the Georgia Milestones webpage at <http://www.gadoe.org/Curriculum-Instruction-and-Assessment/Assessment/Pages/Georgia-Milestones-EOC-Resources.aspx>.

Item 1**Selected-Response**

Sandra makes necklaces and sells them at a school craft fair. She uses the equation $P = 7.5n - (2.25n + 15)$ to determine her total profit at the fair when n necklaces are sold. Based on this equation, how much does she charge for each necklace?

- A. \$2.25
- B. \$7.50
- C. \$15.00
- D. \$17.25

Item 2**Selected-Response**

The perimeter of a rectangle is $P = 2w + 2l$, where w is the width of the rectangle and l is the length of the rectangle. Rearrange this formula to find the width of the rectangle.

- A. $w = P - 2l$
- B. $w = \frac{P}{4 - l}$
- C. $w = 2P - l$
- D. $w = \frac{P}{2} - l$

Item 3**Multi-Select Technology-Enhanced**

The formula shown represents final velocity, v , in terms of initial velocity, v_0 , acceleration, a , and time, t .

$$v = v_0 + at$$

Select **THREE** equations that are equivalent to the final velocity formula.

A. $v_0 = v + at$

B. $v_0 = v - at$

C. $a = \frac{v - v_0}{t}$

D. $a = \frac{v_0 - v}{t}$

E. $t = a(v_0 + v)$

F. $t = \frac{v - v_0}{a}$

Item 4**Drag-and-Drop Technology-Enhanced**

Two inequalities are shown.

Inequality A: $y > -3x + 4$

Inequality B: $y \geq 2x - 1$

Move each coordinate pair to the appropriate column. Coordinate pairs may be used 0 times, 1 time, or 2 times.

?	
Solution to Inequality A	Solution to Inequality B

(1, 2) (-2, -6) (1, 1) (0, 0)



Use a mouse, touchpad, or touchscreen to move coordinate pairs into columns. Each coordinate pair may be used twice.

Item 5

Drag-and-Drop Technology-Enhanced

The steps to correctly solve an equation are shown.

Move a property into each box of the Reason column to justify each step.

	Equation	Reason
Step 1	$18 - 6x = 3(3x - 4)$	original equation
Step 2	$6 - 2x = 3x - 4$	
Step 3	$6 = 5x - 4$	
Step 4	$10 = 5x$	
Step 5	$2 = x$	

Addition Property of Equality	Division Property of Equality
Associative Property	Multiplication Property of Equality
Commutative Property	Reflexive Property of Equality
Distributive Property	Subtraction Property of Equality



Use a mouse, touchpad, or touchscreen to move a property into each blank box. Each property may be used 4 times.

Item 6

Keypad-Input Technology-Enhanced

A table of values is shown.

x	y
0	3
3	5
6	7
9	9

Create a linear equation that represents the values shown in the table.



Use a mouse, touchpad, or touchscreen to enter a response.

Item 7

Drop-Down Multi-Part Technology-Enhanced

Part A

A company sells fruit in cylindrical cans. The cans all have the same height, but there are several different sizes of circular bases. The table shows some information about the different sizes of the cans.

Radius (centimeters)	Area of Base (square centimeters)	Surface Area (square centimeters)	Volume (cubic centimeters)
3	28.26	244.92	282.6
4	50.24	351.68	502.4
5	78.50	471.00	785.0
6	113.04	602.88	1,130.4

Part A The dimensions in the table that have a constant rate relationship are and .

➡ Use a mouse, touchpad, or touchscreen to click the arrow beside each of the two blank boxes. When you click the arrow, a drop-down menu will appear, showing you all the possible options for that blank box. Each drop-down menu with its options is shown below.

The dimensions in the table that have a constant rate relationship are and .

radius
area of base
surface area
volume

radius
area of base
surface area
volume

Go on to the next page to finish item 7.

Item 7. *Continued.*

Part B

A company sells fruit in cylindrical cans. The cans all have the same height, but there are several different sizes of circular bases. The table shows some information about the different sizes of the cans.

Dimensions of Fruit Cans			
Radius (centimeters)	Area of Base (square centimeters)	Surface Area (square centimeters)	Volume (cubic centimeters)
3	28.26	244.92	282.6
4	50.24	351.68	502.4
5	78.50	471.00	785.0
6	113.04	602.88	1,130.4

Part B Which statement explains why these dimensions have a constant rate relationship?

- ☐ (a) Each entry for both dimensions has a factor of 3.14.
- ☐ (b) The second dimension increases by the same percentage over each interval.
- ☐ (c) Dividing the second dimension by the first dimension has the same result for each row.
- ☐ (d) The second dimension can be found by substituting the first dimension into an equation.



Use a mouse, touchpad, or touchscreen to select a response.

Item 8

Drop-Down Technology-Enhanced

Two functions are described below.

- The graph of linear function $f(x)$ has an x -intercept of $(-2, 0)$ and a y -intercept of $(0, 3)$.
- The graph of linear function $g(x)$ is defined by the equation $y = \frac{4}{5}x - 2$.

Use the drop-down menus to complete the statements about the functions.

The y -intercept of the graph of $f(x)$ is the y -intercept of the graph of $g(x)$. The slope of the graph of $f(x)$ is the slope of the graph of $g(x)$.



Use a mouse, touchpad, or touchscreen to click the arrow beside each of the two blank boxes. When you click the arrow, a drop-down menu will appear, showing you all the possible options for that blank box. Each drop-down menu with its options is shown below.

The y -intercept of the graph of $f(x)$ is the y -intercept of the graph of $g(x)$. The slope of the graph of $f(x)$ is the slope of the graph of $g(x)$.

greater than
less than
the same as

greater than
less than
the same as

Item 9**Selected-Response**

Which function can be used to model the data in this table?

x	$f(x)$
0	-1
2	0
6	2

- A. $f(x) = 3x$
- B. $f(x) = \frac{x}{2} - 1$
- C. $f(x) = x - 1$
- D. $f(x) = 2x - 1$

Item 10**Selected-Response**

The function $f(x) = 3x - 9$ is shifted 2 units up. Which equation correctly describes the new function?

- A. $g(x) = 6x - 9$
- B. $g(x) = 3(x + 2) - 9$
- C. $g(x) = 3x - 7$
- D. $g(x) = 6x - 18$

Item 11**Multi-Part Technology-Enhanced**

Tamina drives to a friend's house at an average rate of 45 miles per hour. She wants to determine that rate in feet per minute.

Part A

Which ratio is needed to convert 45 miles per hour into feet per minute?

- A. $\frac{1 \text{ mi}}{5,280 \text{ ft}}$
- B. $\frac{5,280 \text{ ft}}{1 \text{ mi}}$
- C. $\frac{60 \text{ sec}}{1 \text{ min}}$
- D. $\frac{60 \text{ min}}{1 \text{ hr}}$

Part B

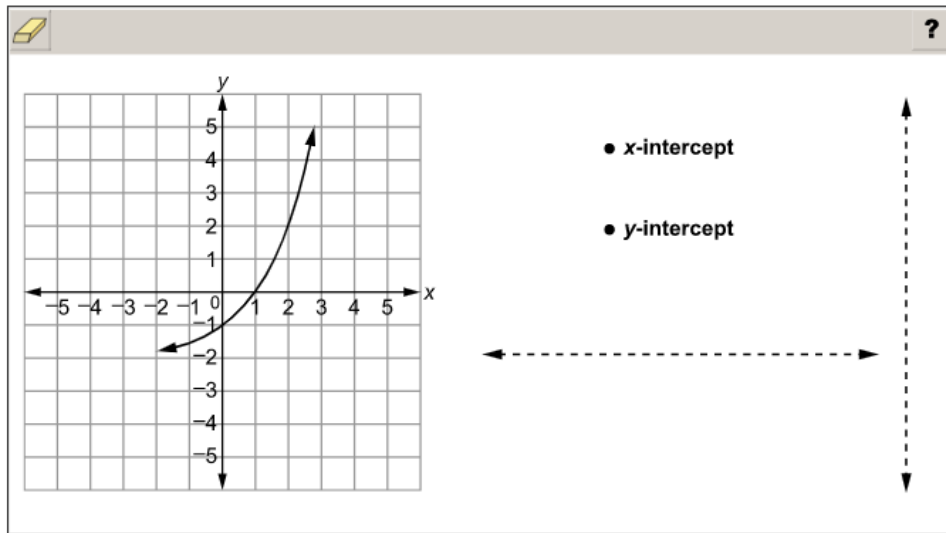
What is 45 miles per hour in feet per minute?

- A. 2,640
- B. 2,700
- C. 3,960
- D. 5,280

Item 12**Drag-and-Drop Technology-Enhanced**

An exponential function is shown.

Move points to represent the x -intercept and the y -intercept. Then move a dotted line to represent the asymptote.



- ➡ Use a mouse, touchpad, or touchscreen to move the labeled points and a dotted line onto the coordinate grid. Each labeled point may be used once.

Item 13**Selected-Response**

A scientist studied the relationship between the number of trees, x , per acre and the number of birds, y , per acre in a neighborhood. She modeled the relationship with a scatter plot and used the equation $y = 4 + 6x$ for the regression line. What is the meaning of the slope and y -intercept of this regression line?

- A. The slope is 6. This means that the average number of birds per acre in an area with no trees is 6. The y -intercept is 4. This means that for every 1 additional tree, she can expect an average of 4 additional birds per acre.
- B. The slope is 4. This means that for every 1 additional tree, she can expect an average of 4 additional birds per acre. The y -intercept is 6. This means that the average number of birds per acre in an area with no trees is 6.
- C. The slope is 6. This means that for every 1 additional tree, she can expect an average of 6 additional birds per acre. The y -intercept is 4. This means that the average number of birds per acre in an area with no trees is 4.
- D. The slope is 4. This means that the average number of birds per acre in an area with no trees is 4. The y -intercept is 6. This means that for every 1 additional tree, she can expect an average of 6 additional birds per acre.

Item 14**Multi-Part Technology-Enhanced**

On Tuesday, Kylie picked raspberries at a farm. The equation shown represents the number of buckets of raspberries she had picked, y , for x hours after 1 p.m.

$$y = \frac{8}{3}x + 7$$

Part A

What does the number $\frac{8}{3}$ represent in the equation?

- A. Kylie can pick 8 buckets of raspberries every 3 hours.
- B. Kylie can pick 3 buckets of raspberries every 8 hours.
- C. At 3 p.m. Kylie had picked a total of 8 buckets of raspberries.
- D. At 8 p.m. Kylie had picked a total of 3 buckets of raspberries.

Part B

What does the number 7 represent in the equation?

- A. Kylie had already picked raspberries for 7 hours.
- B. Kylie had 7 more hours for picking raspberries.
- C. Kylie can pick 7 buckets of raspberries every hour.
- D. Kylie had already picked 7 buckets of raspberries.

Item 15**Selected-Response**

A random group of high school students was surveyed. Each student was asked whether it should be mandatory for all high school students to participate in a sport. The results are partially summarized in the two-way table.

	Agree	Disagree	No Opinion	Total
Freshman	53	12	7	
Sophomore	65	37	2	104
Junior	18	42	12	
Senior	58	67	4	
Total		158		375

What percentage is closest to the number of students in the freshman group who agree that it should be mandatory for all high school students to participate in a sport?

- A. 14.1%
- B. 22.6%
- C. 53%
- D. 73.6%

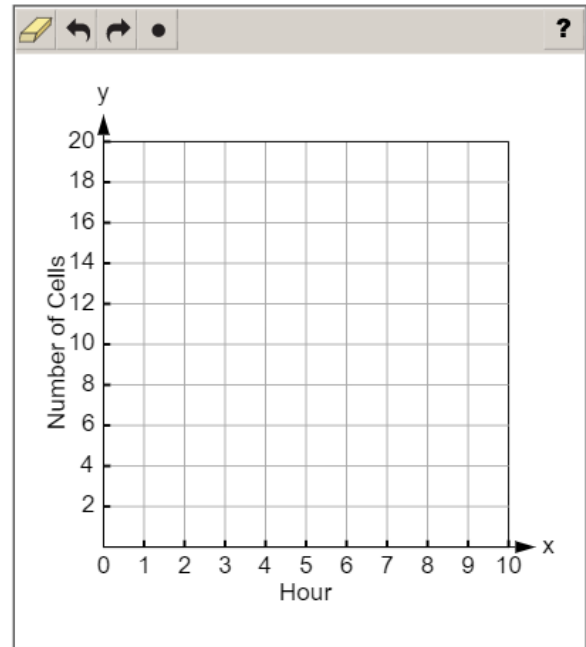
Item 16**Coordinate-Graph Multi-Part Technology-Enhanced****Part A**

A scientist observes cells under a microscope for several hours. The scientist's data are shown in the table.

Hour	0	1	2	3	4
Number of Cells	1	2	4	9	19

Part A

Plot each point on the coordinate grid.



Use a mouse, touchpad, or touchscreen to plot points on the coordinate grid. At most 5 points can be plotted.

Go on to the next page to finish item 16.

Item 16. *Continued.*

Part B

A scientist observes cells under a microscope for several hours. The scientist's data are shown in the table.

Hour	0	1	2	3	4
Number of Cells	1	2	4	9	19

Part B

Which equation BEST fits the data?

☐ a $y = 2x + 1$

☐ b $y = x + 1$

☐ c $y = 3^x$

☐ d $y = 2^x$



Use a mouse, touchpad, or touchscreen to select a response.

Item 17

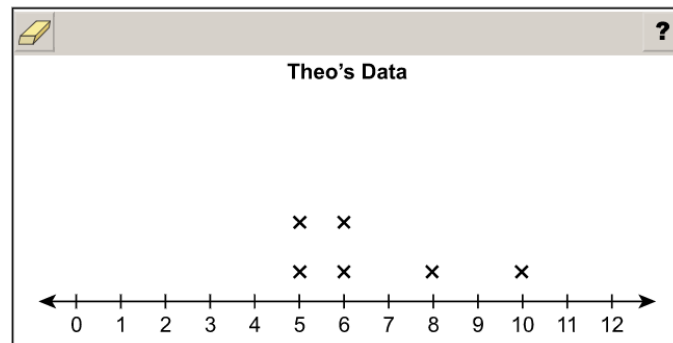
Line-Plot Technology-Enhanced

Kendra and Theo each collected data on the number of birds they observed at a bird feeder each hour for eight hours. The data Kendra collected is summarized as shown.

- The range of Kendra's data is 6.
- The median of Kendra's data is 7.
- The mean of Kendra's data is 7.

The data Theo collected has the same range and the same median as the data Kendra collected, but the mean of Theo's data is 7.5. The line plot correctly shows six of the eight data points for Theo's data set.

Plot the remaining two data points on the line plot.



Use a mouse, touchpad, or touchscreen to add Xs to the line plot. At most 2 Xs can be plotted for each number.

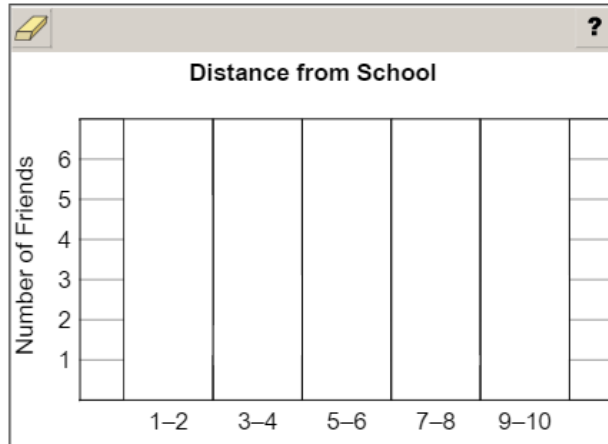
Item 18**Bar-Graph Technology-Enhanced**

Greta collected data on the distance, in miles, that each of her friends lived from their school. The results are shown, but two of the values are missing.

5, 8, 1, 1, 3, 3, ?, ?

When the two missing values are included, the median of the data set will be 3 and the mean of the data set will be 4.

Complete the histogram to represent the data set after determining the two missing values.

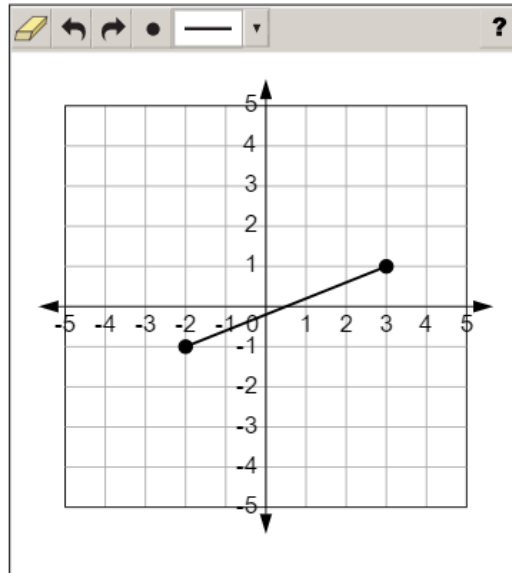


Use a mouse, touchpad, or touchscreen to create each bar in the histogram.

Item 19**Coordinate-Graph Technology-Enhanced**

A line segment is shown on a coordinate plane.

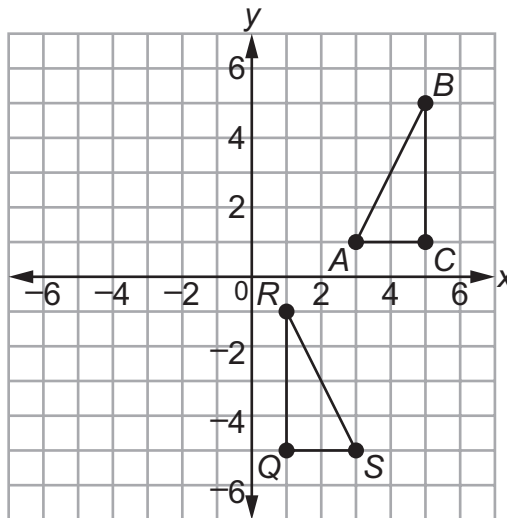
Draw the image of the line segment after a 90° rotation counterclockwise about the point $(-2, -1)$.



Use a mouse, touchpad, or touchscreen to graph a line on the coordinate grid. At most 1 line segment and 2 points can be graphed.

Item 20**Selected-Response**

What is the sequence of transformations that carry triangle ABC to triangle QRS ?



- A. Triangle ABC is reflected across the line $x = 3$. Then it is translated 2 units down.
- B. Triangle ABC is reflected across the line $x = 3$. Then it is translated 6 units down.
- C. Triangle ABC is translated 2 units to the left. Then it is rotated 90 degrees counterclockwise about the point $(1, 1)$.
- D. Triangle ABC is translated 2 units to the right. Then it is rotated 90 degrees counterclockwise about the point $(1, 1)$.

Item 21**Selected-Response**

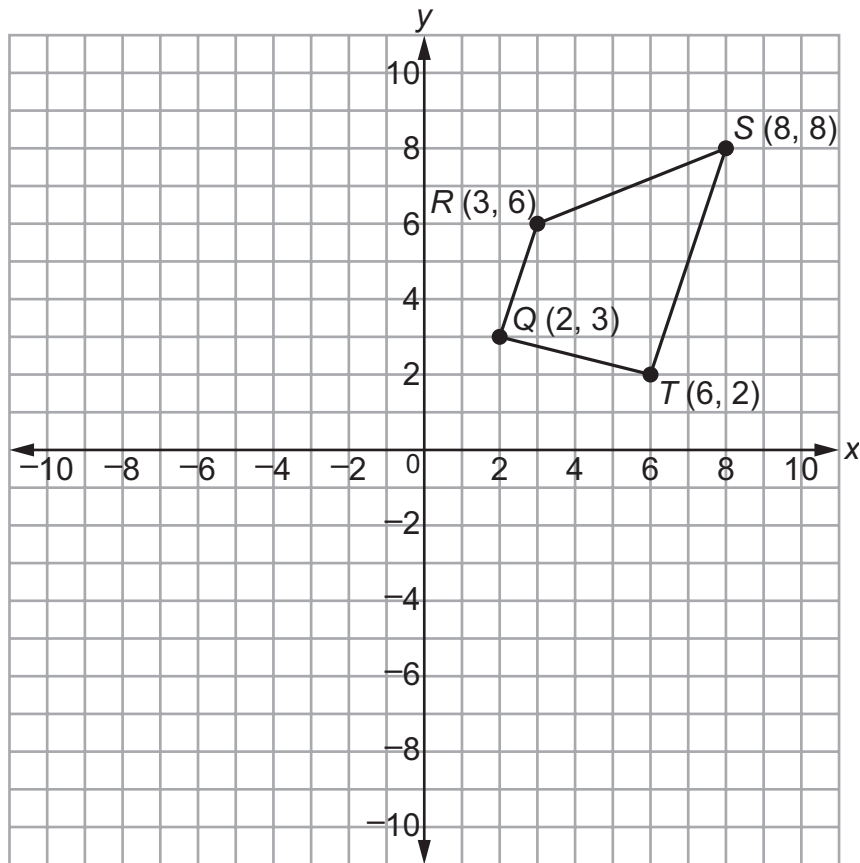
Which transformation on quadrilateral $ABCD$ produces an image that does not preserve distance between points in quadrilateral $ABCD$?

- A. reflection across $y = x$
- B. translation 3 units down and 4 units to the right
- C. dilation by a scale factor of 2
- D. rotation of 270 degrees

Item 22

Selected-Response

Look at quadrilateral $QRST$.

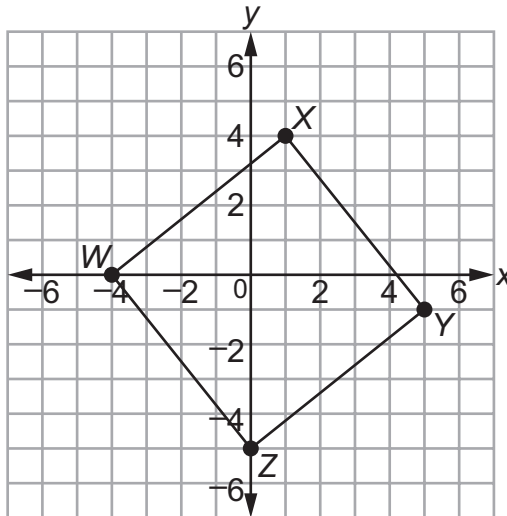


What is the image of point R after a counterclockwise rotation of 270 degrees about the origin?

- A. $(6, -3)$
- B. $(-3, 6)$
- C. $(-6, 3)$
- D. $(3, -6)$

Item 23**Selected-Response**

Look at the square $WXYZ$ on this coordinate plane.



Which measure is closest to the perimeter of square $WXYZ$?

- A. 20 units
- B. 25.6 units
- C. 32 units
- D. 40.9 units

Item 24

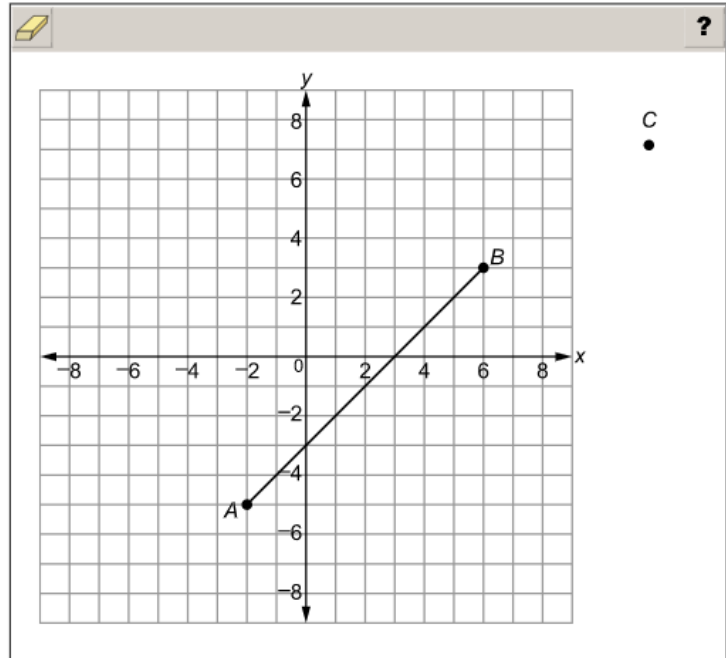
Drag-and-Drop Multi-Part Technology-Enhanced

Part A

Point A is located at $(-2, -5)$. Point B is located at $(6, 3)$. Line segment AB is shown on the coordinate grid.

Part A Point C partitions line segment AB such that the ratio $AC:CB$ is 1:3.

Move point C to plot it in the correct location on the coordinate grid.



Use a mouse, touchpad, or touchscreen to move the labeled point onto the coordinate grid. The labeled point may be used once.

Go on to the next page to finish item 24.

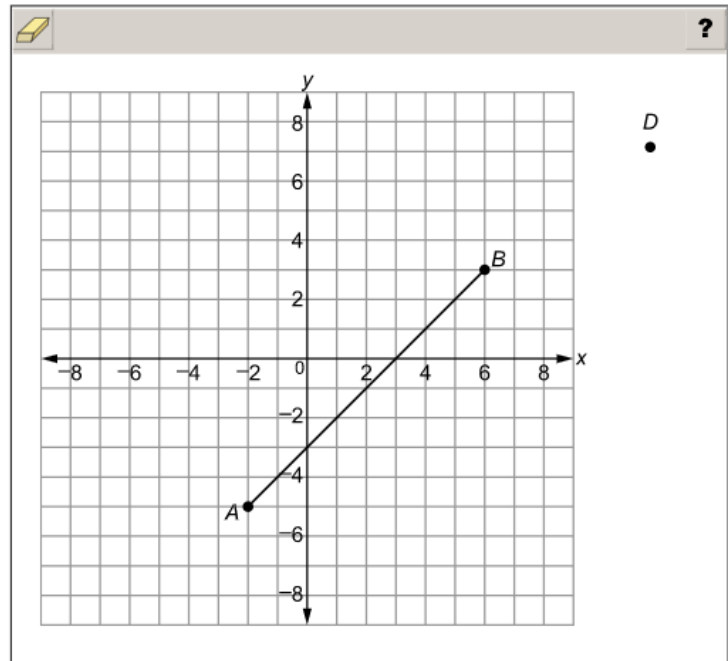
Item 24. *Continued.*

Part B

Point A is located at $(-2, -5)$. Point B is located at $(6, 3)$. Line segment AB is shown on the coordinate grid.

Part B Point D partitions line segment AB such that the ratio $AD:DB$ is $7:1$.

Move point D to plot it in the correct location on the coordinate grid.



- ➡ Use a mouse, touchpad, or touchscreen to move the labeled point onto the coordinate grid. Each labeled point may be used once.

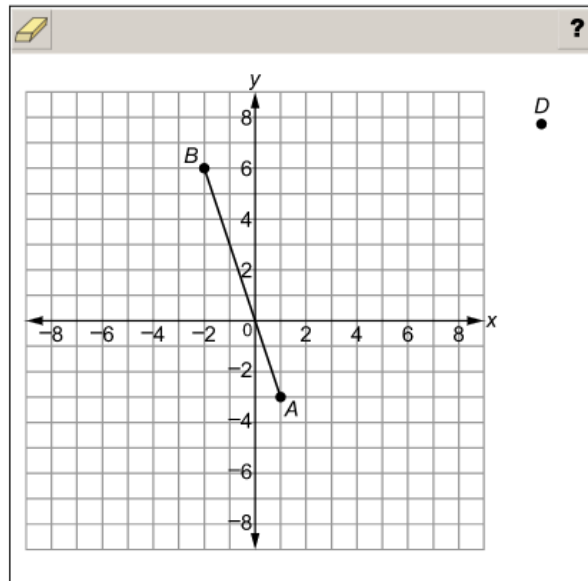
Item 25

Drag-and-Drop Technology-Enhanced

Point A is located at $(1, -3)$. Point B is located at $(-2, 6)$. Line segment AB is shown on the coordinate grid.

Point D partitions line segment AB such that the ratio $AD:DB$ is $7:2$.

Move point D to plot it on the coordinate grid.



Use a mouse, touchpad, or touchscreen to move the labeled point onto the grid. The labeled point may be used once.

Item 26**Selected-Response**

What are the coordinates of a point that lies along the directed line segment from $Q(2, 5)$ to $R(7, 12)$ and partitions the segment in the ratio of 3 to 2?

- A. (3, 4.2)
- B. (4.5, 8.5)
- C. (5, 9.2)
- D. (5, 7)

Item 27**Selected-Response**

What is the equation of the line that is perpendicular to $y = \frac{1}{2}x - 6$ and passes through the point (6, 4)?

- A. $y = -\frac{1}{2}x + 1$
- B. $y = -\frac{1}{2}x + 7$
- C. $y = -2x - 8$
- D. $y = -2x + 16$

ADDITIONAL PRACTICE ITEMS ANSWER KEY

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
1	MGSE9-12.A.SSE.1a	1	B	The correct answer is choice (B) \$7.50. The profit equals 7.5 times the number of necklaces minus the cost per necklace and other fixed costs. Choice (A) is incorrect because it is the cost to produce each necklace. Choice (C) is incorrect because it is a fixed cost. Choice (D) is incorrect because it incorrectly adds the costs.
2	MGSE9-12.A.CED.4	2	D	The correct answer is choice (D) $w = \frac{P}{2} - I$. To isolate w , divide both sides by 2 and then subtract I . Choices (A), (B), and (C) are incorrect because they incorrectly subtracted or divided to rearrange the formula to isolate w .
3	MGSE9-12.A.CED.4	2	B/C/F	The correct answers are choices (B), (C), and (F). Choices (A), (D), and (E) are incorrect because they solve for the terms incorrectly.
4	MGSE9-12.A.CED.3	2	N/A	See scoring rubric and exemplar response on page 219.
5	MGSE9-12.A.REI.1	2	N/A	See scoring rubric and exemplar response on page 220.
6	MGSE9-12.A.CED.2	1	N/A	See scoring rubric and exemplar response on page 221.
7	MGSE9-12.F.LE.1b	2	N/A	See scoring rubric and exemplar response on page 222.
8	MGSE9-12.F.IF.9	2	N/A	See scoring rubric and exemplar response on page 223.
9	MGSE9-12.F.IF.9	1	B	The correct answer is choice (B) $f(x) = \frac{x}{2} - 1$. By substituting the values of x into each function, only this one works for all values of x . Choices (A), (C), and (D) are incorrect because all values of x from the table will not give $f(x)$, even though some of the values might.

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
10	MGSE9-12.F.BF.3	2	D	The correct answer is choice (C) $g(x) = 3x - 7$. A shift up 2 units would add 2 units to the function. Choice (A) is incorrect because the x-value is multiplied by 2. Choice (B) is incorrect because 2 is added to the x-value. Choice (D) is incorrect because the function is multiplied by 2.
11	MGSE9-12.N.Q.1	2	Part A: B Part B: C	Part A: The correct answer is choice (B) $\frac{5,280 \text{ ft}}{1 \text{ mi}}$. This ratio converts miles into feet. Choice (A) is incorrect because it is the reciprocal of the needed conversion. Choice (C) is incorrect because it is the wrong conversion. Choice (D) is incorrect because it is the reciprocal of the other needed conversion. Part B: The correct answer is choice (C) 3,960. The rate of 45 miles per hour is multiplied by 5,280 feet per mile and multiplied by 1 hour per 60 minutes, which is 3,960 feet per minute. Choice (A) is incorrect because it is the result of dividing 5,280 by 2. Choice (B) is incorrect because it is the result of multiplying 60 by 45. Choice (D) is incorrect because it is just the amount of feet per mile.
12	MGSE9-12.F.IF.4	2	N/A	See scoring rubric and exemplar response on page 224.
13	MGSE9-12.S.ID.7	3	C	The correct answer is choice (C). Choice (A) is incorrect because the y-intercept is being used as a rate of change. Choice (B) has the incorrect interpretation of slope within the context. Choice (D) has the incorrect interpretation of slope and y-intercept within the context.

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
14	MGSE9-12.S.ID.7	2	Part A: A Part B: D	<p>Part A: The correct answer is choice (A) Kylie can pick 8 buckets of raspberries every 3 hours. The slope represents how much the y changes per change in x-value, or how many buckets of raspberries are picked per a number of hours. Choice (B) is incorrect because it flips the values of x and y. Choice (C) is incorrect because it mistakes what the y-values represent. Choice (D) is incorrect because it mistakes what the y-values represent and flips the values for x and y.</p> <p>Part B: The correct answer is choice (D) Kylie has already picked 7 buckets of raspberries. The 7 represents the y-intercept, which is the value of the function when x is zero, or at 1 pm. Choices (A), (B), and (C) are incorrect because they misrepresent what the y-intercept means.</p>
15	MGSE9-12.S.ID.5	2	D	The correct answer is choice (D) 73.6%. The conditional relative frequency of freshmen who agree is found by dividing 53 by the total number of freshmen surveyed, 72. Choice (A) is incorrect because 53 was divided by the total number of students surveyed, not just the freshmen. Choice (B) is incorrect because it calculated the ratio of freshmen who agreed to the number of freshmen who disagreed. Choice (C) is incorrect because it is the number, not the percentage, of freshmen who agreed.
16	MGSE9-12.S.ID.6a	2	N/A	See scoring rubric and exemplar response on page 225.
17	MGSE9-12.S.ID.2	3	N/A	See scoring rubric and exemplar response on page 226.
18	MGSE9-12.S.ID.1	2	N/A	See scoring rubric and exemplar response on page 227.
19	MGSE9-12.G.CO.5	2	N/A	See scoring rubric and exemplar response on page 228.
20	MGSE9-12.G.CO.5	1	B	The correct answer is choice (B) Triangle ABC is reflected across the line $x = 3$. Then it is translated 6 units down. Choice (A) is incorrect because the triangle is translated more than 2 units down. Choices (C) and (D) are incorrect because rotating the triangle after the translation would not yield the correct orientation of triangle SRQ .

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
21	MGSE9-12.G.CO.2	1	C	The correct answer is choice (C) dilation by a scale factor of 2. When a figure is dilated, its line segments are either increased or decreased by a scale factor to form a similar figure. Choices (A), (B), and (D) are incorrect because these are rigid transformations that move the figure on the plane without affecting side lengths.
22	MGSE9-12.G.CO.2	2	A	The correct answer is choice (A) (6, -3). Rotating the figure 270 degrees counterclockwise about the origin is the same as rotating it clockwise 90 degrees. Therefore, the figure would be in the 4th quadrant, and point <i>R</i> would be at (6, -3). Choices (B) and (C) are incorrect because they are in the 2nd quadrant. Choice (D) is incorrect because even though it is in the 4th quadrant, the coordinate points are wrong.
23	MGSE9-12.G.GPE.7	2	B	The correct answer is choice (B) 25.6 units. Apply the distance formula to find the length of one side, which is 6.4. Since this is a square, multiply 6.4 by 4 to obtain the perimeter. Choice (A) is incorrect because the number of unit squares on a line segment was counted to estimate the length and then multiplied by 4. Choice (C) is incorrect because the number of unit squares across the square was counted to estimate the length and then multiplied by 2. Choice (D) is incorrect because it is the approximate area of the square.
24	MGSE9-12.G.GPE.6	1	N/A	See scoring rubric and exemplar response beginning on page 229.
25	MGSE9-12.G.GPE.6	1	N/A	See scoring rubric and exemplar response on page 231.
26	MGSE9-12.G.GPE.6	2	C	The correct answer is choice (C) (5, 9.2). The rise from <i>Q</i> to <i>R</i> is 5 and the run from <i>Q</i> to <i>R</i> is 7. Multiply each value by the ratio $\frac{3}{5}$ and then add that amount to the original coordinates (starting point) to find the <i>x</i> - and <i>y</i> -values for point <i>P</i> . Choice (A) is incorrect because these are the <i>x</i> - and <i>y</i> -values that need to be added to point <i>Q</i> . Choice (B) is incorrect because this is the midpoint of the line segment <i>QR</i> . Choice (D) is incorrect because this is the difference of the <i>x</i> -coordinates and <i>y</i> -coordinates.

Item	Standard/ Element	DOK Level	Correct Answer	Explanation
27	MGSE9-12.G.GPE.5	2	D	The correct answer is choice (D) $y = -2x + 16$. This response is correct because the given equation has a slope of $\frac{1}{2}$, and the slope of a perpendicular line will be the negative reciprocal, in this case -2 . This equation will also pass through the point (6, 4). Choices (A) and (B) are incorrect because they do not have slopes that are perpendicular to the given line. Choice (C) is incorrect because the line does not pass through the point (6, 4).

ADDITIONAL PRACTICE ITEMS SCORING RUBRICS AND EXEMPLAR RESPONSES

Item 4

Scoring Rubric

Points	Description
2	The student correctly places coordinate pairs in both columns.
1	The student correctly places coordinate pairs in one of the columns.
0	The student does not correctly place coordinate pairs in either column.

Exemplar Response

The correct response is shown below.

Solution to Inequality A	Solution to Inequality B
(1, 2)	(1, 2) (1, 1) (0, 0)

(1, 2) (-2, -6) (1, 1) (0, 0)

Taking the x - and y -values from each coordinate pair and placing them into the inequalities will determine whether the coordinate pair is a solution to the inequality. If the resulting inequality is true, then the coordinate pair is a solution to the inequality. If the resulting inequality is not true, then the coordinate pair is not a solution to the inequality. The coordinate pair $(1, 2)$ makes true statements in both inequalities, the coordinate pair $(-2, -6)$ does not make a true statement for either inequality, and the coordinate pairs $(1, 1)$ and $(0, 0)$ make only Inequality B true.

Item 5

Scoring Rubric

Points	Description
2	The student correctly places properties in all four rows.
1	The student correctly places properties in two or three of the rows.
0	The student does not correctly place properties in at least two rows.

Exemplar Response

The correct response is shown below.

?

	Equation	Reason
Step 1	$18 - 6x = 3(3x - 4)$	original equation
Step 2	$6 - 2x = 3x - 4$	Division Property of Equality
Step 3	$6 = 5x - 4$	Addition Property of Equality
Step 4	$10 = 5x$	Addition Property of Equality
Step 5	$2 = x$	Division Property of Equality

Addition Property of Equality

Associative Property

Commutative Property

Distributive Property

Division Property of Equality

Multiplication Property of Equality

Reflexive Property of Equality

Subtraction Property of Equality

From Step 1 to Step 2, the division property of equality is used to divide both sides of the equation by 3. From Step 2 to Step 3, the addition property of equality is used to add $2x$ to both sides of the equation. From Step 3 to Step 4, the addition property of equality is used to add 4 to both sides of the equation. From Step 4 to Step 5, the division property of equality is used to divide both sides of the equation by 5.

Item 6

Scoring Rubric

Points	Description
1	The student correctly answers the question.
0	The student does not correctly answer the question.

Exemplar Response

The correct response is shown below.

The screenshot shows a digital math input interface. At the top, there is a toolbar with icons for erasing, undo, redo, left arrow, right arrow, and a close button (X). Below the toolbar, the equation $y = \frac{2}{3}x + 3$ is displayed. Below the equation, there is a keypad with three columns of symbols. The first column contains the numbers 1 through 9, 0, a decimal point, and a negative sign. The second column contains the variables x and y . The third column contains the arithmetic operators $+$, $-$, \cdot , and \div ; the comparison operators $<$, \leq , $=$, \geq , and $>$; and the algebraic symbols $\frac{\square}{\square}$, \square^\square , and (\square) .

This is the correct response because the y -values increase by 2 units for every 3 units the x -values increase by, and y is 3 when x is 0. Any equation that is equivalent to the exemplar response will be given credit.

Item 7**Scoring Rubric**

Points	Description
2	The student correctly answers both Part A and Part B.
1	The student correctly answers either Part A OR Part B.
0	The student does not correctly answer either part.

Exemplar Response**Part A**

The two correct responses are shown below.

The dimensions in the table that have a constant rate relationship are and .

The dimensions in the table that have a constant rate relationship are and .

This is correct because the volume in each row is 10 times the area of the base in that row. These dimensions are the only ones that exhibit a linear relationship.

Part B

The correct answer is choice (C) Dividing the second dimension by the first dimension has the same result for each row. This shows that the volume increases by the same amount over equal intervals of the area of the base, which is necessary for a linear relationship. Choice (A) is incorrect because factors of the dimensions are not related to their linear relationship; other dimensions also have a factor of 3.14. Choice (B) is incorrect because constant rate relationships increase by the same amount over equal intervals rather than the same percentage. Choice (D) is incorrect because substitution can be used in relationships that are not linear.

Item 8**Scoring Rubric**

Points	Description
1	The student correctly selects both drop-down menu options.
0	The student does not correctly select both drop-down menu options.

Exemplar Response

The correct response is shown below.

The y-intercept of the graph of $f(x)$ is the y-intercept of the graph of $g(x)$. The slope of the graph of $f(x)$ is the slope of the graph of $g(x)$.

“Greater than” is the correct response for the first drop-down menu because the y-intercept of $f(x)$ is 3 and the y-intercept of $g(x)$ is -2 . “Greater than” is the correct response for the second drop-down menu because the slope of $f(x)$ is 1.5 and the slope of $g(x)$ is 0.8.

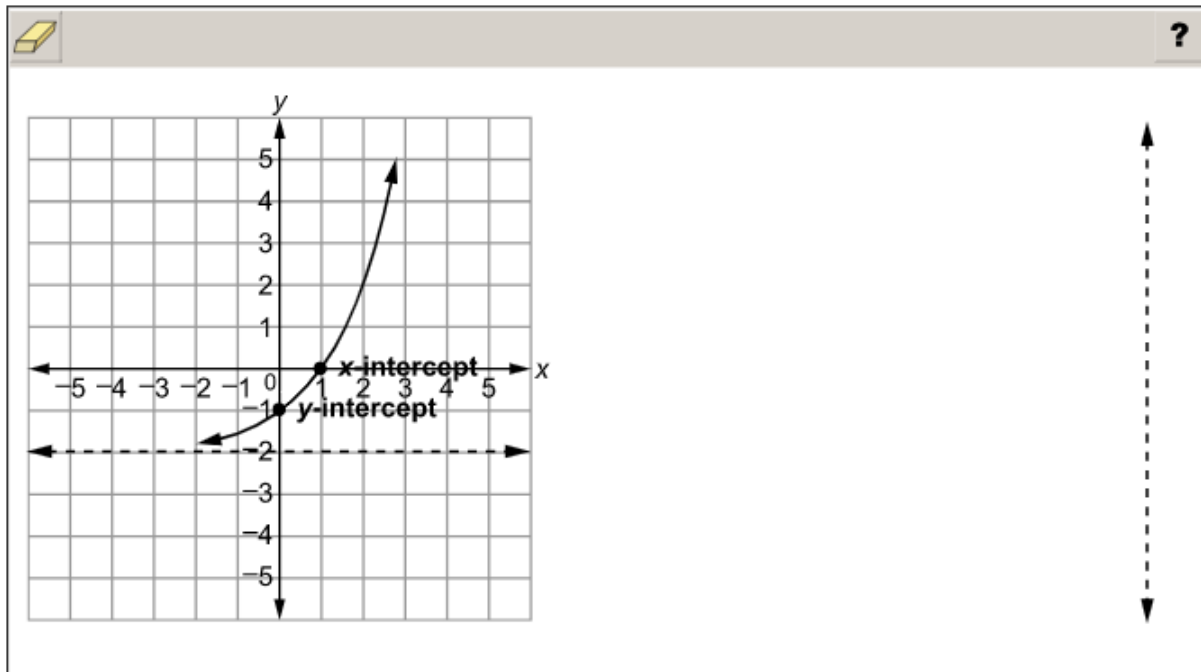
Item 12

Scoring Rubric

Points	Description
2	The student correctly places the x-intercept, y-intercept, and horizontal asymptote.
1	The student correctly places the x-intercept and y-intercept or the student correctly places the horizontal asymptote.
0	The student does not correctly place the x-intercept and y-intercept or the horizontal asymptote.

Exemplar Response

The correct response is shown below.



Exponential functions have horizontal asymptotes. The asymptote of this function is $y = -2$. The y-intercept is the point at which the graph crosses the y-axis, and the x-intercept is the point at which the graph crosses the x-axis.

Item 16

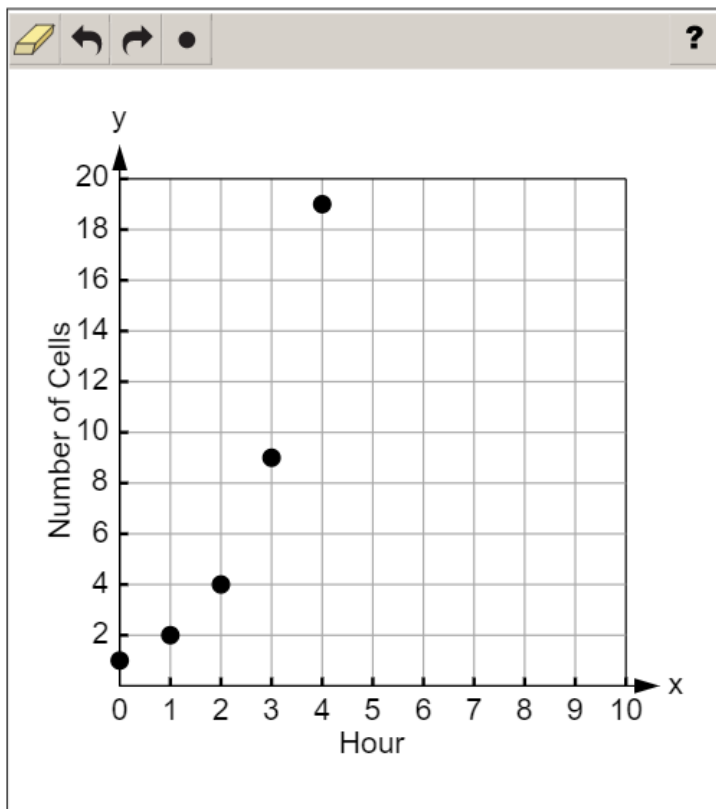
Scoring Rubric

Points	Description
2	The student correctly answers both Part A and Part B.
1	The student correctly answers either Part A OR Part B.
0	The student does not correctly answer either part.

Exemplar Response

Part A

The correct response is shown below.



The points from the table are represented by the coordinate pairs (0, 1), (1, 2), (2, 4), (3, 9), and (4, 19).

Part B

The correct answer is choice (D) $y = 2^x$. The information in the table represents a function that is increasing at an increasing rate, which is an exponential function. At $x = 1$, the value of the function is 2; therefore, $y = 2^x$ is the exponential function that passes through this point. Choice (A) is incorrect because it represents a linear graph. Choice (B) is incorrect because it represents a linear graph. Choice (C) is incorrect because it represents an exponential function that passes through the coordinate pair (1, 3) and not the coordinate pair (1, 2).

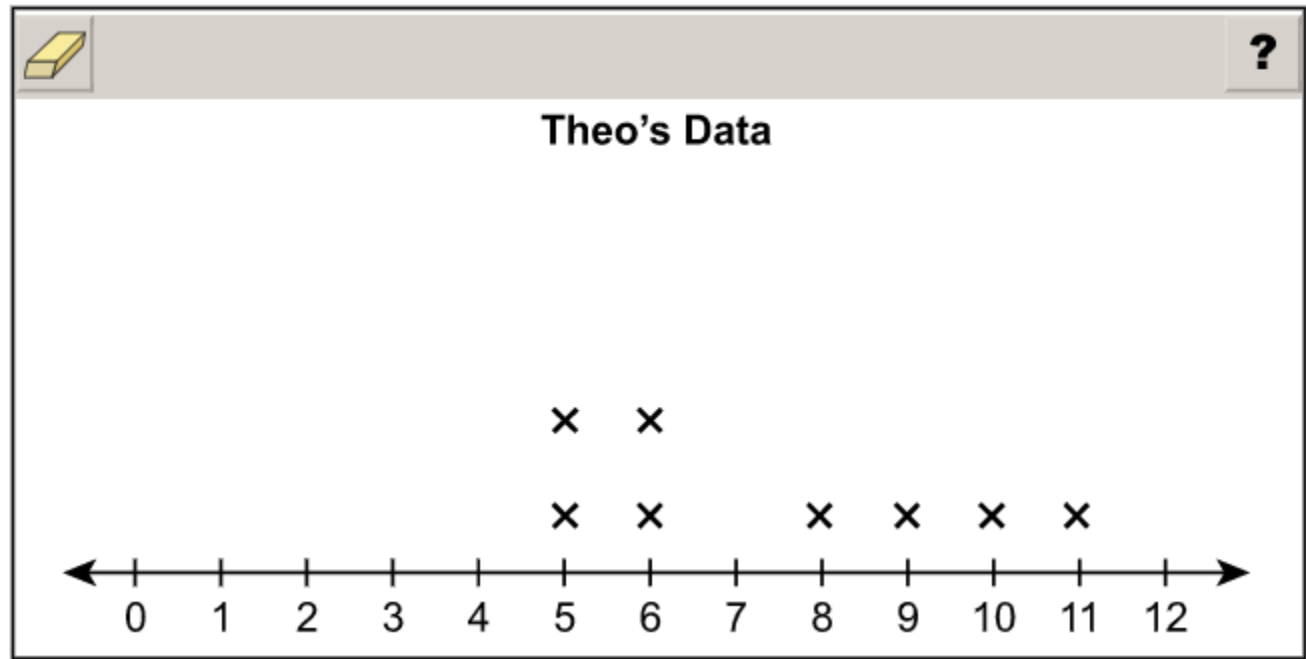
Item 17

Scoring Rubric

Points	Description
1	The student correctly plots the two Xs.
0	The student does not correctly plot the two Xs.

Exemplar Response

The correct response is shown below.



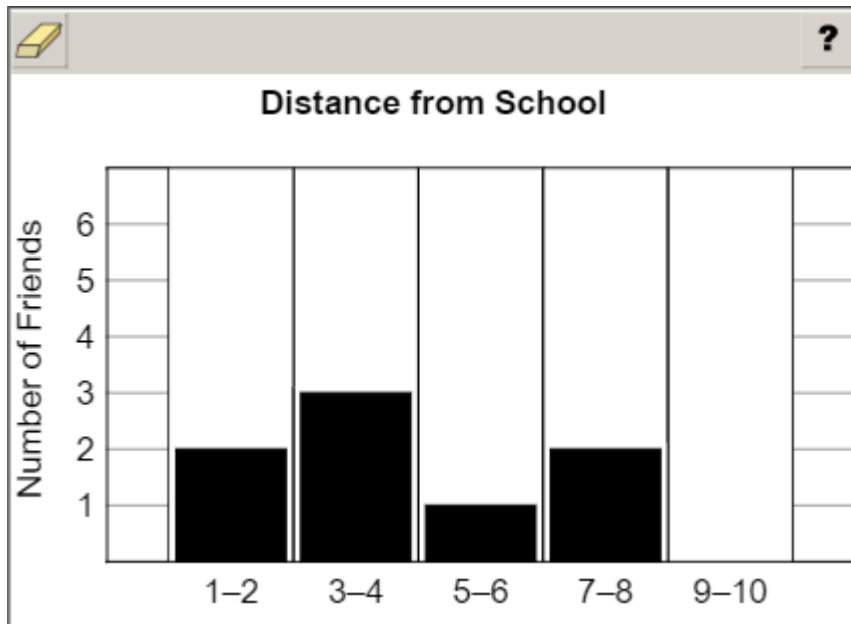
This is the correct response because in order for Theo's data set to have a range of 6, a median of 7, and a mean of 7.5, the two data points must be plotted at 9 and 11.

Item 18**Scoring Rubric**

Points	Description
1	The student correctly completes the histogram.
0	The student does not correctly complete the histogram.

Exemplar Response

The correct response is shown below.



This is the correct response because the two missing numbers are 3 and 8. Adding those two values to the data set will result in the median being 3 and the mean being 4.

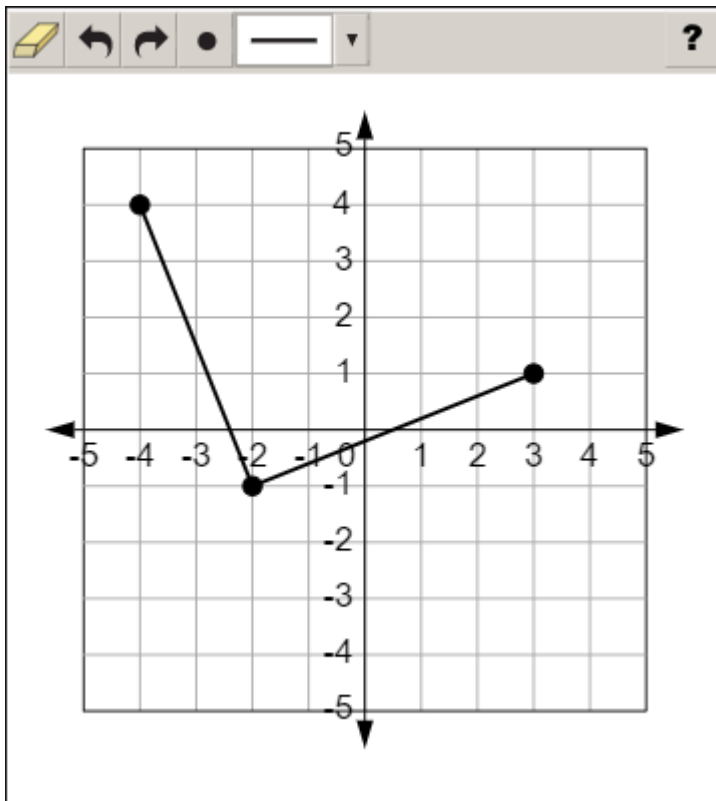
Item 19

Scoring Rubric

Points	Description
1	The student correctly graphs the line segment.
0	The student does not correctly graph the line segment.

Exemplar Response

The correct response is shown below.



This is the correct response because rotating a line segment 90 degrees about one of its endpoints results in a right angle. Therefore, the slopes of the line segments must be opposite reciprocals. The slope of the original line segment is $-\frac{5}{2}$, and the slope of the rotated line segment is $\frac{2}{5}$. The line segments are also congruent because rotations preserve length.

Item 24

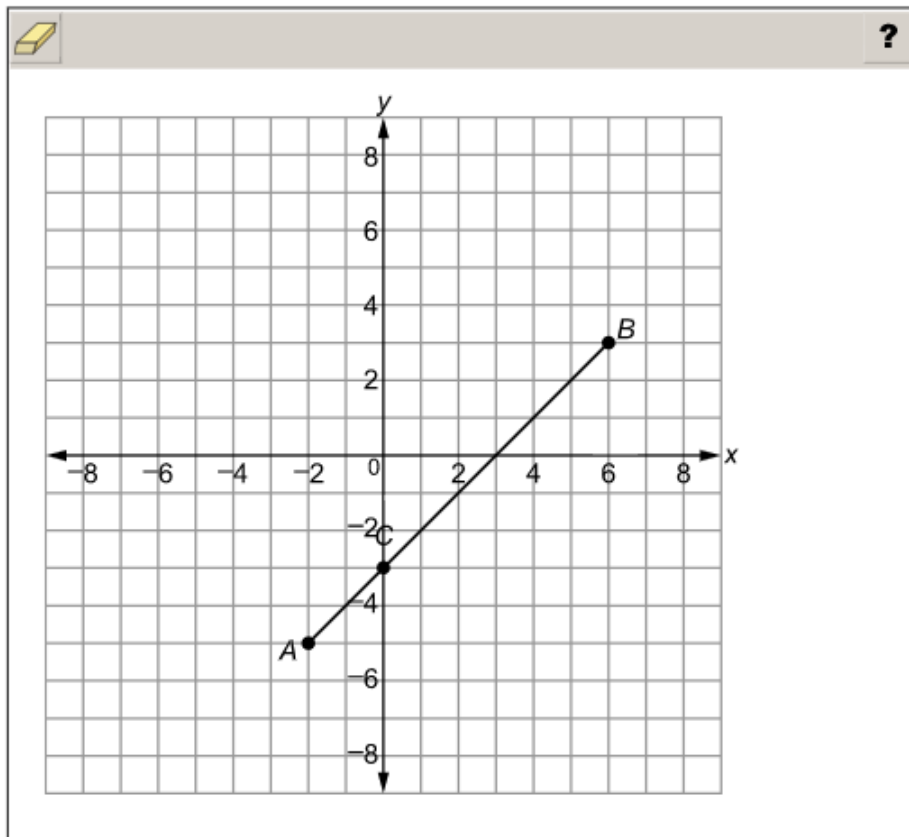
Scoring Rubric

Points	Description
2	The student correctly answers both Part A and Part B.
1	The student correctly answers either Part A OR Part B.
0	The student does not correctly answer either part.

Exemplar Response

Part A

The correct response is shown below.

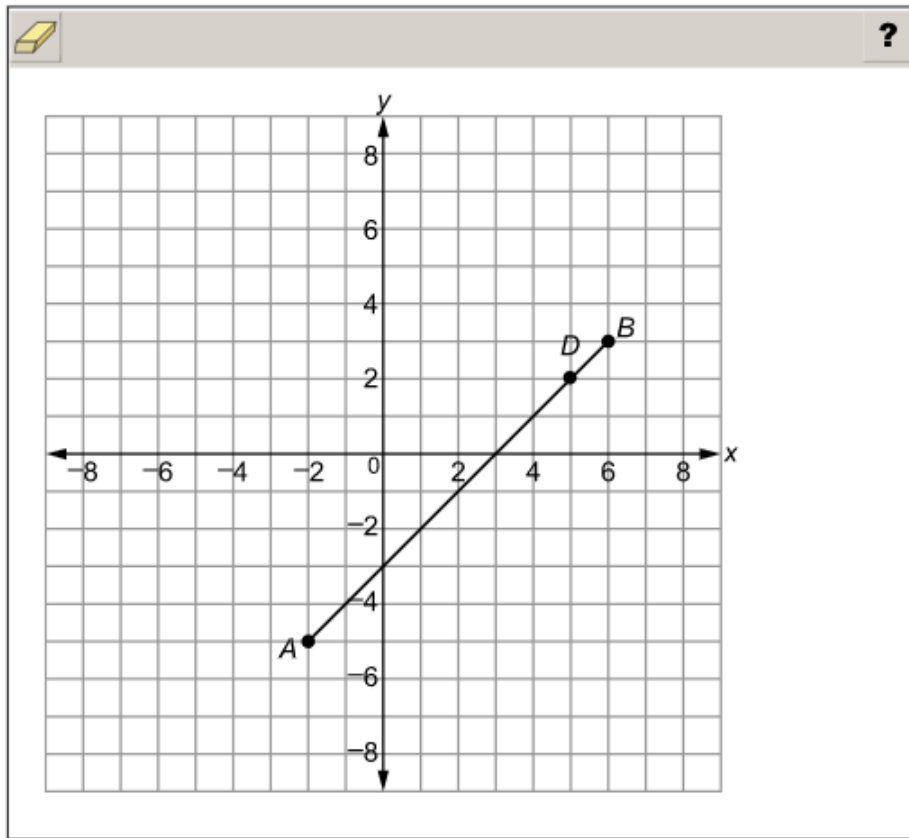


Use the formula on the given formula sheet for partitioning a line segment to determine the location of point C. In the formula, $a = 1$, $b = 3$, $x_1 = -2$, $y_1 = -5$, $x_2 = 6$, and $y_2 = 3$. Solving the formula gives the location $(0, -3)$ for point C.

Go on to the next page to finish item 24.

Item 24**Part B**

The correct response is shown below.



Use the formula on the given formula sheet for partitioning a line segment to determine the location of point D. In the formula, $a = 7$, $b = 1$, $x_1 = -2$, $y_1 = -5$, $x_2 = 6$, and $y_2 = 3$. Solving the formula gives the location (5, 2) for point D.

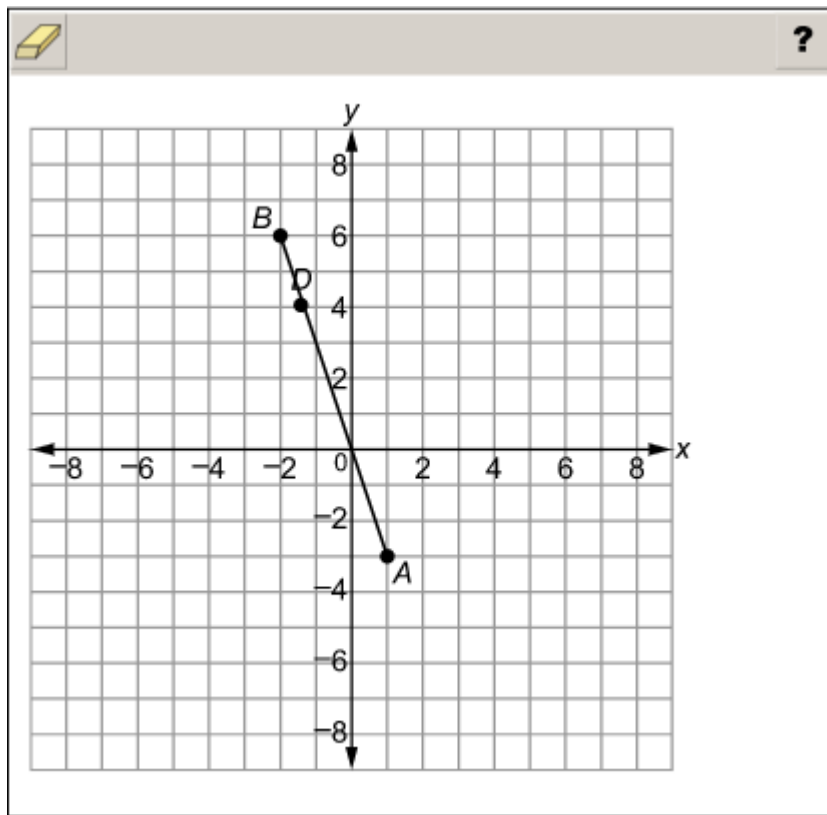
Item 25

Scoring Rubric

Points	Description
1	The student correctly places the point.
0	The student does not correctly place the point.

Exemplar Response

The correct response is shown below.



In the formula for partitioning a line segment substitute the following values, $a = 7$, $b = 2$, $x_1 = 1$, $y_1 = -3$, $x_2 = -2$, and $y_2 = 6$, to determine the location of point D. Solving the formula gives the location $(-\frac{4}{3}, 4)$ for point D.

Study/Resource
Guide for Students
and Parents
Coordinate Algebra
End-of-Course

